

SEMI-ALGEBRAIC HORIZONTAL SUBVARIETIES OF CALABI–YAU TYPE

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ABSTRACT. We study horizontal subvarieties Z of a Griffiths period domain \mathbf{D} . If Z is defined by algebraic equations, and if Z is also invariant under a large discrete subgroup in an appropriate sense, we prove that Z is a Hermitian symmetric domain \mathcal{D} , embedded via a totally geodesic embedding in \mathbf{D} . Next we discuss the case when Z is in addition of Calabi–Yau type. We classify the possible VHS of Calabi–Yau type parametrized by Hermitian symmetric domains \mathcal{D} and show that they are essentially those found by Gross and Sheng–Zuo, up to taking factors of symmetric powers and certain shift operations. In the weight three case, we explicitly describe the embedding $Z \hookrightarrow \mathbf{D}$ from the perspective of Griffiths transversality and relate this description to the Harish-Chandra realization of \mathcal{D} and to the Korányi–Wolf tube domain description. There are further connections to homogeneous Legendrian varieties and the four Severi varieties of Zak.

INTRODUCTION

Let \mathbf{D} be a period domain, i.e. a classifying space for polarized Hodge structures of weight n with Hodge numbers $\{h^{p,q}\}$, $p + q = n$. It has been known since Griffiths’ pioneering work that, unless \mathbf{D} is a classifying space for weight one Hodge structures (polarized abelian varieties) or weight two Hodge structures satisfying $h^{2,0} = 1$, then most of the points of \mathbf{D} do not come from algebraic geometry in any sense. More precisely, any geometrically defined variation of Hodge structure is contained in a horizontal subvariety of \mathbf{D} , an integral manifold for the differential system corresponding to Griffiths transversality, and the union of all such arising from algebraic geometry is a countable union of proper subvarieties of \mathbf{D} . It is thus of interest to write down specific examples of such horizontal subvarieties Z . One of the simplest cases where Griffiths transversality is a nontrivial condition is the case of weight three Hodge structures of Calabi–Yau type, i.e. $h^{3,0} = 1$. This case is also very important for geometric reasons.

In the general case, since \mathbf{D} is an open subset of its compact dual $\check{\mathbf{D}}$, and $\check{\mathbf{D}}$ is a projective variety, it is natural to look at those Z which can be defined algebraically, i.e. such that Z is a connected component of $\hat{Z} \cap \mathbf{D}$, where \hat{Z} is a closed algebraic subvariety of $\check{\mathbf{D}}$. We will refer to such Z as *semi-algebraic in \mathbf{D}* .

A closed horizontal subvariety Z of \mathbf{D} coming from algebraic geometry satisfies an additional condition: if Γ is the stabilizer of Z in the natural arithmetic group $\mathbf{G}(\mathbb{Z})$ acting on \mathbf{D} , then Γ acts properly discontinuously on Z , the image $\Gamma \backslash Z \subseteq \mathbf{G}(\mathbb{Z}) \backslash \mathbf{D}$ is a closed subvariety, and it is the image of a quasi-projective variety under a proper holomorphic map. Thus it is reasonable in general to look at closed horizontal

Date: September 27, 2011.

The second author was partially supported by NSF grant DMS-0968968.

subvarieties Z of \mathbf{D} such that $\Gamma \backslash Z$ is quasi-projective. Actually, we will need a mild technical strengthening of this hypothesis which we call *strongly quasi-projective* (Definition 1.2). Under this hypothesis, we prove the following theorem in Section 1:

Theorem 1. *Let Z be a closed horizontal subvariety of a classifying space \mathbf{D} for Hodge structures and let Γ be the stabilizer of Z in the appropriate arithmetic group $\mathbf{G}(\mathbb{Z})$. Assume that*

- (i) $S = \Gamma \backslash Z$ is strongly quasi-projective;
- (ii) Z is semi-algebraic in \mathbf{D} .

Then Z is a Hermitian symmetric domain whose embedding in \mathbf{D} is an equivariant, holomorphic, horizontal embedding.

The main ingredients used in the proof are the theorem of the fixed part as proved by Schmid for variations of Hodge structure over quasi-projective varieties, Deligne's characterization of Hermitian symmetric domains [Del79], and the recent theory of Mumford–Tate domains as developed by Green–Griffiths–Kerr [GGK11]. Theorem 1, in the case where \mathbf{D} itself is Hermitian symmetric (and thus S is a subvariety of a Shimura variety), has been proved independently by Ullmo–Yafaev [UY11], using similar methods. This result is related in spirit, but in a somewhat different direction, to a conjecture of Kollár [KP11], which says roughly that, if Z is simply connected and semi-algebraic, and Z/Γ is projective for some discrete group Γ of biholomorphisms of Z , then Z is the product of a Hermitian symmetric space and a simply connected projective variety.

The remainder of the paper is concerned with Hodge structures of *Calabi–Yau type*, in other words effective weight n Hodge structures such that $h^{n,0} = 1$. For Hermitian symmetric spaces of tube type, Gross [Gro94] has constructed certain natural variations of Hodge structure of Calabi–Yau type. This construction was extended by Sheng–Zuo [SZ10] to the non-tube case to construct complex variations of Hodge structure. Their methods can easily be adapted to construct real variations of Hodge structure of Calabi–Yau type. In Section 2, we show that all real variations of Hodge structure over a Hermitian symmetric space can be constructed via standard techniques from those of Gross and Sheng–Zuo. In the spirit of Gross, we classify real variations of Hodge structure or \mathbb{Q} -variations of Hodge structure which remain irreducible over \mathbb{R} . More precisely, we show:

Theorem 2. *For every irreducible Hermitian symmetric domain of non-compact type $\mathcal{D} = G(\mathbb{R})/K$, there exists a canonical \mathbb{R} -variation of Hodge structure \mathcal{V} of Calabi–Yau type. Any other irreducible equivariant \mathbb{R} -variation of Hodge structure of Calabi–Yau type on \mathcal{D} can be obtained as a summand of $\mathrm{Sym}^n \mathcal{V}$ or $\mathrm{Sym}^n \mathcal{V} \{ -\frac{a}{2} \}$ (if \mathcal{D} is not a tube domain), where $\{ \}$ denotes the half-twist operation (cf. [vG01]).*

The case of weight three is described in detail in e.g. Corollary 2.34. In Section 3, we make some remarks about the more difficult problem of classifying irreducible \mathbb{Q} -variations of Hodge structure of Calabi–Yau type.

In Sections 4 and 5, we specialize further to the case of weight three and discuss various methods for constructing semi-algebraic variations of Hodge structure, not necessarily strongly quasi-projective. The method of Section 4 gives a construction of maximal horizontal subvarieties (see (4.12)) adapted to finding a rational or real structure on the group and in particular to finding a unipotent arithmetic group action on the horizontal subvariety whose general elements are maximally unipotent. These considerations lead to a homogeneous cubic polynomial $\varphi(z_1, \dots, z_h)$,

where h is the dimension of the horizontal subvariety. However, for a general cubic polynomial φ , the horizontal subvarieties so constructed will have no symplectic automorphisms other than the unipotent group (Theorem 4.16) and so will not be strongly quasi-projective. Additionally, we discuss the Hodge–Riemann bilinear relation in terms of the cubic φ (Theorem 4.18). In Section 5, we give a related construction which leads to complex polynomials analogous to φ , as well as a variant which leads to non-maximal variations (see (5.2) and (5.5) respectively), which are relevant in the non-tube case.

In Section 6, we show that the weight three Hermitian symmetric examples can be described by the methods of Section 5 in general (Theorem 6.10) and by those of Section 4 in the tube domain case (Theorem 6.5). Perhaps not surprisingly, the realization of Hermitian symmetric domains as horizontal subvarieties of the CY period domain is closely related to the general theory of realizations of these symmetric domains. Roughly speaking, the methods of Section 4 correspond to the unbounded realizations of Hermitian symmetric spaces due to Korányi–Wolf [KW65], while those of Section 5 correspond to the Harish-Chandra embedding of a Hermitian symmetric space as a bounded domain. It is very likely that similar explicit constructions can describe the general weight n case. However, the case of weight three, aside from being the simplest nontrivial case, also has many connections with other geometric questions. For example, over \mathbb{C} , the classification in the tube domain case is equivalent to the theory of homogeneous Legendrian varieties as studied extensively by Landsberg–Manivel (e.g. [LM07], [LM01]). It is also related to Zak’s classification of Severi varieties (see for instance [LVdV84]). Finally, both of the exceptional Hermitian symmetric domains (namely type EIII and EVII) appear as horizontal subvarieties of weight three variations of Hodge structure of Calabi–Yau type.

Lastly, in Section 7, we describe some of the interesting Hodge theory in the weight three tube domain case in terms of the cubic form φ as well as the Hermitian symmetric space structure. In particular we analyze (1) the locus where the intermediate Jacobian of the weight three Hodge structure is an almost direct product, where one factor is a polarized abelian variety, and (2) degenerations and limiting mixed Hodge structures. The discussion is not meant to be definitive in any sense.

Although there is a great deal of literature on the subject, we are not concerned in this paper with realizing the Hermitian symmetric examples geometrically, i.e. as variations of Hodge structure associated to a family of Calabi–Yau or other smooth projective varieties or via some motivic construction beginning with such a family. Some of the known Hermitian type examples of geometric nature include those constructed by Borcea [Bor97] and Voisin [Voi93], and the more recent examples of ball quotient type due to Rohde [Roh09] and van Geemen and his coauthors [GvG10] (for further discussion see Remark 2.13). Such examples tend to be quite rare: the horizontal subvarieties associated to most geometric examples are far from being equivariantly embedded Hermitian symmetric. For example, if the Zariski closure of the monodromy group is the full symplectic group, or contains monodromy transformations of Picard–Lefschetz type (see for example Definition 7.10), then the horizontal subvariety is not a Hermitian symmetric space equivariantly embedded in the classifying space. This fact rules out the quintic threefold and its mirror and tends to rule out most complete intersections in toric varieties as well. Similarly,

Gerkmann et al. [GSvSZ10] show that the moduli space of Calabi–Yau threefolds obtained via double-covers of \mathbb{P}^3 branched in 8 general planes is not Hermitian symmetric.

Convention: We abbreviate variation of Hodge structure by VHS. All VHS are polarizable/polarized, and satisfy the Griffiths transversality condition. A Hodge structure or VHS (of weight k) will be assumed to be effective ($h^{p,q} \neq 0$ only for $p, q \geq 0$, $h^{k,0} \neq 0$) unless otherwise noted. Of course, by a Tate twist, we can always arrange that a Hodge structure is effective. We denote by \mathbf{D} a Griffiths period domain. Thus, $\mathbf{D} = \mathbf{G}(\mathbb{R})/\mathbf{K}$, where \mathbf{G} is an orthogonal or symplectic group defined over \mathbb{Q} (the group preserving a pair (V, Q) , where V is a \mathbb{Q} -vector space and Q is a non-degenerate symmetric or alternating form defined over \mathbb{Q}) and \mathbf{K} is a compact subgroup of $\mathbf{G}(\mathbb{R})$, not in general maximal.

1. SEMI-ALGEBRAIC IMPLIES HERMITIAN SYMMETRIC

Our goal in this section is to prove Theorem 1.

Definition 1.1. Let $\mathbf{D} = \mathbf{G}(\mathbb{R})/\mathbf{K}$ be a classifying space for Hodge structures with compact dual $\check{\mathbf{D}} = \mathbf{G}(\mathbb{C})/\mathbf{P}(\mathbb{C})$. A closed horizontal subvariety Z of \mathbf{D} will be called *semi-algebraic in \mathbf{D}* if Z is an open subset of its Zariski closure $\hat{Z} \subseteq \check{\mathbf{D}}$. Equivalently, there exists a closed subvariety \hat{Z} of the projective variety $\check{\mathbf{D}}$ such that Z is a connected component of $\hat{Z} \cap \mathbf{D}$. Note that, if Z is semi-algebraic in \mathbf{D} , then Z is a semi-algebraic set.

Definition 1.2. Let $\mathbf{D} = \mathbf{G}(\mathbb{R})/\mathbf{K}$ be a classifying space for Hodge structures as above, and let Z be a closed horizontal subvariety of \mathbf{D} . Let $\Gamma = \Gamma_Z$ be the stabilizer of Z in $\mathbf{G}(\mathbb{Z})$, i.e.

$$\Gamma = \{\gamma \in \mathbf{G}(\mathbb{Z}) : \gamma(Z) = Z\}.$$

Thus Γ acts properly discontinuously on Z . We call $\Gamma \backslash Z$ *strongly quasi-projective* if, for every subgroup Γ' of Γ of finite index, the analytic space $\Gamma' \backslash Z$ is quasi-projective, and thus the morphism $\Gamma' \backslash Z \rightarrow \Gamma \backslash Z$ is a morphism of quasi-projective varieties. In particular, if $\Gamma \backslash Z$ is strongly quasi-projective, then $\Gamma \backslash Z$ is quasi-projective.

- Remark 1.3.* (i) If Γ acts on Z without fixed points and $\Gamma \backslash Z$ is quasi-projective, then by the Riemann’s existence theorem $\Gamma \backslash Z$ is automatically strongly quasi-projective.
- (ii) If \mathbf{D} is Hermitian symmetric, so that the quotient of \mathbf{D} by every arithmetic subgroup admits a Baily-Borel compactification, then it is easy to check that $\Gamma \backslash Z$ is strongly quasi-projective if and only if $\Gamma \backslash Z$ is quasi-projective.

We can now restate Theorem 1 as follows:

Theorem 1.4. *Let Z be a closed horizontal subvariety of a classifying space $\mathbf{D} = \mathbf{G}(\mathbb{R})/\mathbf{K}$ for Hodge structures and let $\Gamma = \text{Stab}_Z \cap \mathbf{G}(\mathbb{Z})$ as above. Assume that*

- (i) $\Gamma \backslash Z$ is strongly quasi-projective;
- (ii) Z is semi-algebraic in \mathbf{D} .

Then Z is a Hermitian symmetric domain $G(\mathbb{R})/K$, whose embedding in \mathbf{D} is an equivariant, holomorphic, horizontal embedding. In other words, G is a closed algebraic subgroup of \mathbf{G} defined over \mathbb{Q} , $K = \mathbf{K} \cap G(\mathbb{R})$ is a maximal compact subgroup, the complex structures on $Z = G(\mathbb{R})/K$ and \mathbf{D} are induced by a morphism $S^1 \rightarrow G(\mathbb{R}) \subseteq \mathbf{G}(\mathbb{R})$, and $Z = G(\mathbb{R})/K$ is a horizontal submanifold of \mathbf{D} .

Proof. The embedding $Z \subseteq \mathbf{D}$, induces a variation of Hodge structures \mathcal{V} on $S := \Gamma \backslash Z$ with associated monodromy group Γ . Consider the generic Mumford–Tate group M associated to \mathcal{V} and the derived subgroup $M' = M_{der}$. Let $M_1 = \overline{\Gamma}^0$ be the connected component of the Zariski closure of the monodromy group Γ . Note that M_1 is an algebraic subgroup of \mathbf{G} such that $M_1(\mathbb{C})$ is invariant under $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and hence M_1 is defined over \mathbb{Q} . By a theorem of Deligne (see e.g. [Mil11b, Theorem 6.19(a)]), $M_1 \subseteq M'$. Since S is quasi-projective, a theorem of André ([And92, Theorem 1]) implies that M_1 is in fact a normal subgroup of M' . Since M' is semi-simple, M' is an almost direct product of M_1 and another semi-simple subgroup $M_2 \subseteq M'$, isogenous to M'/M_1 . Without loss of generality, after possibly by passing to a subgroup of Γ of finite index, we can assume that

- i) $\Gamma \subseteq M_1(\mathbb{C}) \cap \mathbf{G}(\mathbb{Z})$ is a subgroup of finite index, and
- ii) $\Gamma \cap M_2(\mathbb{C}) = \{1\}$.

By construction, Γ is a Zariski dense arithmetic subgroup of $M_1(\mathbb{Z})$. Note also that the real group $M_1(\mathbb{R})$ is the intersection of $M_1(\mathbb{C})$ with $\mathbf{G}(\mathbb{R})$.

Let $D_M \subseteq \mathbf{D}$ be the associated Mumford–Tate subdomain, i.e. $D_M = M' \cdot z_0$, the M' -orbit of a general point $z_0 \in Z$ (see [GGK11, Chapter II.B, especially p. 45]). Then D_M is a homogeneous complex submanifold of \mathbf{D} . Moreover, the Zariski closure of D_M in $\check{\mathbf{D}} = \mathbf{G}(\mathbb{C})/\mathbf{P}(\mathbb{C})$ is the homogeneous space

$$\check{D}_M = M'(\mathbb{C})/(M'(\mathbb{C}) \cap \mathbf{P}(\mathbb{C})) = M'(\mathbb{C}) \cdot z_0,$$

and D_M is an open subset of \check{D}_M ([GGK11, Remark on p. 46], [GGK11, Prop. VI.B.11]). Since M is the generic Mumford–Tate group, it follows that $Z \subseteq D_M$ ([Mil11b, §6.15], [GGK11, Theorem III.A.1 (Step 3)]). Clearly, the Zariski closure \hat{Z} of Z in $\check{\mathbf{D}}$ will then satisfy $\hat{Z} \subseteq \check{D}_M$.

Let $z_0 \in Z \subseteq D_M$ be a general reference Hodge structure. Since $M_1 \times M_2 \rightarrow M'$ is an isogeny, at the level of Lie algebras we have $\mathfrak{m}' = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. For the reference Hodge structure on \mathfrak{m}' induced by z_0 , the subalgebras \mathfrak{m}_1 and \mathfrak{m}_2 are sub-Hodge structures. Define $D_{M_i} = M_i \cdot z_0 \subseteq \mathbf{D}$ and similarly for $\check{D}_{M_i} \subseteq \check{\mathbf{D}}$. By arguing as in the proof of [GGK11, Theorem III.A.1 (Step 3)] (see also [GGK11, Theorem IV.F.1]), we conclude that the spaces D_{M_i} and \check{D}_{M_i} are complex submanifolds of $\check{\mathbf{D}}$ and that the natural holomorphic map $\check{D}_{M_1} \times \check{D}_{M_2} \rightarrow \check{D}_M$ is a finite covering space. By passing to the adjoint forms of M and hence M_1 and M_2 , we may further assume that $\check{D}_{M_1} \times \check{D}_{M_2} \cong \check{D}_M$, and similarly that $D_{M_1} \times D_{M_2} \cong D_M$. This has the effect of replacing Z (and \hat{Z}) by a finite quotient by a subgroup of the center of M_1 , and it is clearly enough to prove the theorem for such a finite quotient.

We first claim that the algebraic assumption on Z gives

$$(1.5) \quad \hat{Z} \cap \check{D}_{M_1} = \check{D}_{M_1}.$$

In fact, $\hat{Z}' := \hat{Z} \cap \check{D}_{M_1}$ is an intersection of two closed algebraic subvarieties in $\check{\mathbf{D}}$, and thus it is a closed algebraic subvariety in $\check{\mathbf{D}}$. By assumption, $\Gamma \cdot Z = Z$. Hence $\Gamma \cdot \hat{Z}' = \hat{Z}'$ and so Γ preserves \hat{Z}' . Since \hat{Z}' is algebraic, the Zariski closure $\overline{\Gamma}^0(\mathbb{C})$ also stabilizes \hat{Z}' . Since $\overline{\Gamma}^0(\mathbb{C}) = M_1(\mathbb{C})$ acts transitively on \check{D}_{M_1} , the equality (1.5) follows.

By taking the intersection with \mathbf{D} , (1.5) implies that

$$(1.6) \quad D_{M_1} \subseteq Z,$$

or at least that a connected component of D_{M_1} is contained in Z . Hence the induced variation of Hodge structure on D_{M_1} satisfies Griffiths transversality. It then follows from a result of Deligne [Del79] that D_{M_1} is a Hermitian symmetric domain (with a totally geodesic embedding in \mathbf{D}) (see also [Mil11b, Theorem 7.9]). To complete the proof, we will show that equality holds in (1.6).

Note that $Z \subseteq D_{M_1} \times D_{M_2}$ induces a projection morphism

$$\pi: \Gamma \backslash Z \rightarrow \Gamma \backslash D_{M_1}.$$

Since Γ is an arithmetic subgroup of M_1 , $\Gamma \backslash D_{M_1}$ is quasi-projective (by the Baily–Borel theorem) and the morphism π is algebraic (by Borel’s extension theorem). Furthermore, (1.6) gives a section of this map over the connected component of $\Gamma \backslash D_{M_1}$ containing the image of π .

Let F be a fiber of the morphism $\pi: \Gamma \backslash Z \rightarrow \Gamma \backslash D_{M_1}$. By the discussion of the previous paragraph, F is a quasi-projective variety. On the other hand, $F = Z \cap (\{z_1\} \times D_{M_2})$ for some $z_1 \in D_{M_1}$. We conclude that F is quasi-projective and carries a VHS with trivial monodromy (since $\Gamma \cap M_2(\mathbb{C}) = \{1\}$, since we have replaced Γ by a suitable subgroup of finite index). The theorem of the fixed part as proved by Schmid then implies that the VHS on F is constant. Hence F is a finite set of points. We conclude that $\pi: \Gamma \backslash Z \rightarrow \Gamma \backslash D_{M_1}$ is a quasi-finite map of quasi-projective varieties. It also has a section given by (1.6). But then D_{M_1} is a component of Z . Since Z was assumed irreducible, it follows that $Z = D_{M_1}$ is a Hermitian symmetric domain as described in the statement of Theorem 1.4. \square

Remark 1.7. (i) It is natural to ask if it is sufficient to assume only that $\Gamma \backslash Z$ is quasi-projective.

(ii) More generally, suppose that, instead of assuming that $\Gamma \backslash Z$ is strongly quasi-projective, we assume $\Gamma \backslash Z$ is *strongly compactifiable*, in other words that for every subgroup Γ' of finite index in Γ , $\Gamma' \backslash Z$ can be embedded in a compact analytic space as the complement of an analytic subspace, in such a way that the morphism $\Gamma' \backslash Z \rightarrow \Gamma \backslash Z$ extends to a meromorphic map on the compactifications. The proof of Theorem 1.4 then carries over to this case given the full strength of Schmid’s result, that the theorem of the fixed part holds in this case and noting that André’s theorem on the monodromy only uses the fixed part assumption (see [Mil11b, Theorem 6.19 ii])). More precisely, in this case Z is Hermitian symmetric, Γ is an arithmetic subgroup of the corresponding Lie group $G(\mathbb{R})$, the quotient $\Gamma \backslash Z$ is quasi-projective and a compactification of $\Gamma \backslash Z$ as above is unique up to bimeromorphic equivalence (via Borel’s extension theorem).

(iii) There are also variants of Theorem 1.4 where we assume that $\Gamma \backslash Z$ is the image of a quasi-projective variety S under a proper morphism, such that all finite covers $S \times_{\Gamma \backslash \mathbf{D}} (\Gamma' \backslash \mathbf{D})$ are quasi-projective, and similarly where we just assume that S and its related finite covers are compactifiable.

(iv) Finally, one can ask if Theorem 1.4 extends to the case where $\Gamma \backslash Z$ is only assumed to have finite volume.

2. CLASSIFICATION OF THE HERMITIAN VHS OF CALABI–YAU TYPE

In this section, we classify the VHS parametrized by Hermitian symmetric domains as in Theorem 1.4 for Hodge structures of CY type (Definition 2.7). A similar classification problem is the classical Shimura case, namely the classification of VHS

of weight 1 (polarized abelian varieties) parametrized by Hermitian symmetric domains due to Satake [Sat65] and Deligne [Del79, §1.3] (see also [Mil11b, Chapter 10]). Some examples of Hermitian VHS of CY type were constructed by Gross [Gro94] and Sheng–Zuo [SZ10]. Here, we complete the classification in the CY case (Theorem 2.11 and Corollary 2.34 for weight 3) by showing that all Hermitian VHS of CY type are derived from the examples of Gross and Sheng–Zuo.

Definition 2.1. Let \mathbf{D} be a classifying space of (polarized) Hodge structures. We say that a horizontal subvariety $\mathcal{D} \hookrightarrow \mathbf{D}$ is of *Hermitian type* if \mathcal{D} is a Hermitian symmetric domain of non-compact type embedded in \mathbf{D} via an equivariant (or, equivalently, totally geodesic), holomorphic, horizontal embedding. When $\mathcal{D} \subset \mathbf{D}$ is of Hermitian type, the induced variation of Hodge structures \mathcal{V} on \mathcal{D} is called a *Hermitian VHS*.

Remark 2.2. The Hermitian VHS are the VHS parametrized by Hermitian symmetric domains considered by Deligne [Del79]. Also, in the terminology of [GGK11], a subvariety $\mathcal{D} \subset \mathbf{D}$ of Hermitian type is the same thing as a Mumford–Tate domain which is Hermitian symmetric and unconstrained.

Let $\mathcal{D} = G(\mathbb{R})/K$ be a Hermitian symmetric domain, where G is simple and simply connected real algebraic group and K a maximal compact subgroup. We denote by $\bar{G} = G/Z(G)$ the adjoint form of G , and by $\bar{K} \subset \bar{G}$ the corresponding compact subgroup. The choice of a reference point $z_0 \in \mathcal{D}$ determines the compact subgroup \bar{K} with center S^1 . In particular, the choice of a reference point gives a cocharacter:

$$(2.3) \quad \varphi: S^1 \rightarrow \bar{G}.$$

A holomorphic horizontal embedding $\mathcal{D} \subset \mathbf{D}$ as in Theorem 1.4 is induced by a representation (defined over \mathbb{Q})

$$(2.4) \quad \rho: G \rightarrow \mathrm{GL}(V)$$

and a compatible polarization Q on V , so that $\rho(G) \subseteq \mathbf{G} = \mathrm{Aut}(V, Q)$. For our purposes, there is no loss of generality in assuming that V is irreducible over \mathbb{Q} . In what follows, we will assume that V remains an irreducible representation over \mathbb{R} as well. We will discuss the general case in §3.

Convention 2.5. $G(\mathbb{R})$ is a simple and simply connected algebraic group of Hermitian type. We assume that the group G and the representation ρ are defined over \mathbb{Q} and that the representation $\rho: G(\mathbb{R}) \rightarrow \mathrm{GL}(V_{\mathbb{R}})$ is irreducible.

Following Deligne [Del79] (see also [Mil11b, Chapter 10] and [GGK11, Chapter IV]) in order for the representation ρ to arise from a VHS, there exists a reductive group $M \subset \mathrm{GL}(V)$ defined over \mathbb{Q} (i.e. the generic Mumford–Tate group of the VHS) and a morphism of algebraic groups:

$$h: \mathbb{S} \rightarrow M(\mathbb{R}) \subset \mathrm{GL}(V_{\mathbb{R}})$$

(where $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ is the Deligne torus) such that:

- i) h defines a Hodge structure on V .
- ii) The representation ρ factors through M and $\rho(G) = M_{\mathrm{der}}$.
- iii) The induced morphism

$$(2.6) \quad \bar{h}: \mathbb{S}/\mathbb{G}_m \rightarrow M_{\mathrm{ad}}(\mathbb{R}) = \bar{G}$$

is conjugate to the cocharacter $\varphi: S^1 \rightarrow \bar{G}$.

We are concerned here with the classification of Hermitian VHS of CY type, by which we understand:

Definition 2.7. A *Hodge structure* V of *Calabi–Yau (CY) type* is an effective weight n Hodge structure such that $V^{n,0}$ is 1-dimensional. If $n = 1$, we say that V is of *elliptic curve type*, and if $n = 2$ we say that V is of *K3 type*.

Thus, to classify Hermitian VHS of CY type over $\mathcal{D} = G(\mathbb{R})/K$ (under the assumption 2.5) is equivalent to finding representations V of G and a lifting $h: \mathbb{S} \rightarrow M$ of φ such that the CY condition is satisfied. Typically a compatible polarization exists and it is unique (see [GGK11, §IV.A, (Step 4)]); thus the issue of polarization does not play a major role in the classification.

Gross [Gro94] has constructed \mathbb{R} -VHS of CY type on each of the Hermitian symmetric domains that are of tube type. More precisely, as recalled in §2.1, a Hermitian symmetric domain $\mathcal{D} = G(\mathbb{R})/K$ corresponds to the choice of a root system R and of a minuscule weight ϖ_i . In the tube case, the fundamental representation V_{ϖ_i} associated to ϖ_i is a representation of real type for $G(\mathbb{R})$ (see Lemma 2.21). By considering the representation $\rho: G \rightarrow \mathrm{GL}(V_{\varpi_i})$ and the group $M = G \cdot \mathbb{G}_m$ (where $\mathbb{G}_m \subset \mathrm{GL}(V_{\varpi_i})$ are the homotheties), one obtains a Hermitian VHS of CY type on \mathcal{D} . In the non-tube domain case, the representation V_{ϖ_i} of $G(\mathbb{R})$ is of complex type. Sheng–Zuo [SZ10] showed that this representation leads to a \mathbb{C} -VHS of CY type on the Hermitian symmetric domains of non-tube type. The examples of [SZ10] can be naturally modified to give rise to a \mathbb{R} -VHS: Given a \mathbb{R} -Hodge structure V and a decomposition $V_{\mathbb{C}} = V_+ \oplus V_-$ such that $\overline{V_+} = V_-$, one obtains \mathbb{C} -Hodge structures on V_{\pm} that are conjugate (i.e. $\overline{V_+^{p,q}} = V_-^{q,p}$). Conversely, given conjugate \mathbb{C} -Hodge structures V_+ and V_- , their direct sum $V_+ \oplus V_-$ is naturally an \mathbb{R} -Hodge structure.

A decomposition of type $V_{\mathbb{C}} = V_+ \oplus V_-$ occurs in the presence of *weak CM* (see [vG01, §1.2] and [GGK11, Definition V.B.1(i)]). Specifically, given a Hodge structure V of weight n with weak CM by a CM field F (and a choice of CM type for F), there is a natural (eigenspace) decomposition $V_{\mathbb{C}} = V_+ \oplus V_-$ such that $\overline{V_+} = V_-$. Furthermore, van Geemen [vG01] noted that in this situation the Hodge structure on V can be modified by a *half twist*. Given $a \in \mathbb{Z}$, we define $V\{-\frac{a}{2}\}$ to be the (not necessarily effective) Hodge structure of weight $n + a$ obtained by shifting the weights of V_{\pm} (or equivalently twisting by a certain character) as follows:

$$(2.8) \quad V\left\{-\frac{a}{2}\right\} := V_+ \left\langle -\frac{a}{2} \right| \oplus V_- \left| -\frac{a}{2} \right\rangle,$$

where for a \mathbb{C} -Hodge structure $W = \bigoplus_{p+q=k} W^{p,q}$, we define the left and right shifts by

$$(2.9) \quad W\langle c \rangle := \bigoplus_{(p-2c)+q=k-2c} W^{p-2c,q}, \text{ and resp. } W|a \rangle := \bigoplus_{p+(q-2c)=k-2c} W^{p,q-2c},$$

i.e. $(W\langle c \rangle)^{r,s} = W^{r+2c,s}$ and $(W|c \rangle)^{r,s} = W^{r,s+2c}$ (see also [vG01, §1.4] and [GGK11, Definition V.B.1(v)]). The *half twist* then corresponds to the case $a = 1$. For a general a , positive or negative, we shall refer to $V\{-\frac{a}{2}\}$ as an *iterated half-twist*. Returning to our situation, we will see that the examples of Sheng–Zuo [SZ10] can be interpreted as \mathbb{R} -VHS with weak CM (by an imaginary quadratic field). Thus, one can understand the Hermitian VHS of CY type of Gross [Gro94] and Sheng–Zuo [SZ10] in a uniform way.

Remark 2.10. One should not confuse $V\{c\}$ (or the shifts $\langle |$ and $| \rangle$) with the Tate twist $V(c)$. In fact, applying twice a half-twist does not give a Tate twist, but applying a half-twist, followed by complex conjugation (i.e. switch V_+ and V_-), followed by another half-twist, is the same as a Tate twist. Also note that the half-twist $\{ \}$ and the Tate twist $()$ are applied to \mathbb{R} -Hodge structures, while the shifts $\langle |$ and $| \rangle$ are only applied to \mathbb{C} -Hodge structures. For all four operations, the sign convention is the same as that for Tate twist: the twist by $c \in \mathbb{Z}$ (or $\frac{1}{2}\mathbb{Z}$ respectively) decreases the weight by $2c$ for $\{ \}$, $()$, $\langle |$, and $| \rangle$.

With these preliminaries, we can state the main result of the section:

Theorem 2.11. *For every irreducible Hermitian symmetric domain \mathcal{D} , there exists a canonical \mathbb{R} -variation of Hodge structures \mathcal{V} of CY type parametrized by \mathcal{D} . The VHS \mathcal{V} is uniquely determined by the requirement that it have minimal weight. Specifically,*

- i) *if \mathcal{D} is of tube domain type, \mathcal{V} is the variation constructed by Gross [Gro94], and its weight is equal to the real rank of \mathcal{D} ;*
- ii) *if \mathcal{D} is not of tube domain type, \mathcal{V} is obtained by taking the sum*

$$(\mathcal{W}\langle c|) \oplus (\mathcal{W}^\vee|c)$$

of the \mathbb{C} -VHS \mathcal{W} constructed by Sheng–Zuo [SZ10] with the dual VHS \mathcal{W}^\vee and applying an appropriate twist. The minimal weight giving a VHS of CY type is equal the real rank of \mathcal{D} plus 1.

Any other irreducible \mathbb{R} -VHS of CY type on \mathcal{D} can be obtained from the canonical \mathcal{V} by taking the unique irreducible factor of $\mathrm{Sym}^n \mathcal{V}$ of CY type, or, in the non-tube domain case, by taking the unique irreducible factor of $\mathrm{Sym}^n \mathcal{V}\{-\frac{a}{2}\}$ of CY type for appropriate integers a .

The proof of the theorem is given in §2.1. A detailed discussion of the representations that occur is in §2.2. We will discuss some related rationality questions in Section 3.

Remark 2.12. If \mathcal{D} is a product $\mathcal{D}_1 \times \mathcal{D}_2$ of Hermitian symmetric domains, then a Hermitian VHS \mathcal{V} on \mathcal{D} decomposes as a product $\mathcal{V}_1 \otimes \mathcal{V}_2$. It is clear that \mathcal{V} is of CY type iff each \mathcal{V}_i is of CY type.

Remark 2.13 (Hermitian VHS of geometric origin). We list some Hermitian VHS of CY type that arise in a geometric context:

- (1) (Type III _{n}) The middle cohomology of abelian n -folds gives a weight n VHS of CY type parametrized by the Siegel upper half space \mathfrak{H}_n .
- (2) (Type I_{1,1} \times IV _{n}) Borcea [Bor97] and Voisin [Voi93] have constructed families of Calabi–Yau threefolds such that the associated VHS is parametrized by $\mathfrak{H} \times \mathcal{D}$, where \mathfrak{H} is the upper half plane and \mathcal{D} is a Type IV domain. The Calabi–Yau threefolds are obtained by resolving $(E \times S)/\langle \tau \rangle$, where E is an elliptic curve, S is a $K3$ surface and τ an involution (acting diagonally).
- (3) (Type I_{1, n}) A slight modification of the Borcea–Voisin construction, by considering higher order automorphisms, leads to VHS associated to Calabi–Yau threefolds that are parametrized by complex balls \mathcal{B}_n (e.g. Rohde [Roh09], and Garbagnati–van Geemen [GvG10]).
- (4) (Type Sym^2 IV _{n}) It is interesting to note that cases of type $\mathrm{Sym}^n V$ occur in geometric situations. We thank K. O’Grady for informing us of the

following example (unpublished). Let $X \subset \mathbb{P}^5$ be a generic EPW sextic (cf. [O'G06]). It is known that X is singular along a surface S and that the resolution \tilde{X} is a Calabi–Yau fourfold. On the other hand, there exists a smooth double cover $Y \xrightarrow{2:1} X$ branched only along $S \subset X$ which is a hyperkähler manifold (cf. [O'G06]). It is well known that $V = H_0^2(Y)$ is of $K3$ type, leading to a VHS \mathcal{V} parametrized by a 20-dimensional type IV domain \mathcal{D} . On the other hand, the primitive cohomology groups $H_0^4(\tilde{X})$ and $H_0^4(Y)$ are isogenous, and $H_0^4(Y)$ is essentially $\text{Sym}^2 H_0^2(Y)$. Thus, the periods of the Calabi–Yau 4-folds \tilde{X} are parametrized by Hermitian symmetric domain \mathcal{D} , and the horizontal embedding $\mathcal{D} \hookrightarrow \mathbf{D}$ corresponds to the VHS $\text{Sym}^2 \mathcal{V}$.

2.1. Proof of Theorem 2.11. We fix $\mathcal{D} = G(\mathbb{R})/K$ an irreducible Hermitian symmetric domain of non-compact type, a reference point z_0 (generic with respect to the \mathbb{Q} -structure), and the corresponding cocharacter $\varphi: S^1 \rightarrow \bar{G}$. Let \mathfrak{g} be the corresponding (real) Lie algebra, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition, and H_0 the differential of φ . Note that H_0 spans the center \mathfrak{z} of \mathfrak{k} , \mathfrak{p} is identified with the tangent space of \mathcal{D} at z_0 , and the $\text{ad}(H_0)$ action on \mathfrak{p} gives the complex structure (e.g. [Kna02, Theorem 7.117]). We fix a compact Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{k}$ with $H_0 \in \mathfrak{h}_0$. Let $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C} \subset \mathfrak{g}_{\mathbb{C}}$ be the Cartan subalgebra, and let R be the associated root system. In this situation, the roots are either compact or non-compact. Fix a good ordering of the roots (e.g. [Kna02, p. 441]), and let $\alpha_1, \dots, \alpha_d$ be the simple roots. Then there exist a single simple non-compact root α_i (e.g. [Kna02, p. 449]). Thus, up to a factor of $2\pi\sqrt{-1}$ that we ignore in what follows,

$$(2.14) \quad \begin{cases} \alpha_i(H_0) = 1, \\ \alpha_j(H_0) = 0, \quad \text{if } j \neq i \end{cases}$$

for simple roots (N.B. $\alpha(H_0) = 0$ iff α is compact). The root α_i is a *special root*, i.e. it occurs with multiplicity 1 in the highest root $\tilde{\alpha}$. In fact, the choice of \mathcal{D} is equivalent to a root system R and the choice of a special root α_i (e.g. (R, α_i)) gives the Vogan diagram of real Lie algebra \mathfrak{g} ; see also [Mil11b, Chapter 2]). The possibilities are listed in Table 1. The choice of root α_i singles out the fundamental representation V_{ϖ_i} of highest weight ϖ_i of $G(\mathbb{C})$, and in fact the representations V_{ϖ_i} that occur are precisely the minuscule representations.

Remark 2.15. The relevance of the minuscule weights in the classification of Hermitian symmetric domains can be understood also as follows: The non-compact Hermitian symmetric domains $\mathcal{D} = G(\mathbb{R})/K$ are in one to one correspondence with their compact duals $\tilde{\mathcal{D}} = G(\mathbb{C})/P$. Then, the minimal homogeneous embedding $\tilde{\mathcal{D}}$ in a projective space is given by $\tilde{\mathcal{D}} = G(\mathbb{C}) \cdot [v] \hookrightarrow \mathbb{P}(V_{\varpi_i})$, where v is a highest weight vector in the minuscule representation V_{ϖ_i} .

Our goal is to classify the Hermitian VHS \mathcal{V} of CY type over \mathcal{D} (under the assumption 2.5). As explained above, we consider a representation

$$\rho: G \rightarrow \text{GL}(V)$$

that factors through a reductive group M with $M_{\text{der}} = \rho(G)$, and a lift h of φ . Since $\rho(G) \subset \text{SL}(V)$, it suffices to restrict to the subgroup $\text{Hg} = M \cap \text{SL}(V)$ (thought of as the generic Hodge group of \mathcal{V}) and to understand the lifting of φ to $h|_{\text{U}(1)}$, where $\text{U}(1) \hookrightarrow \mathbb{S}$ is the kernel of the norm map $\mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \rightarrow \mathbb{G}_m$. In other words,

the Hodge decomposition on $V_{\mathbb{C}}$ is the weight decomposition of $V_{\mathbb{C}}$ with respect to $U(1, \mathbb{C}) \cong \mathbb{C}^*$: $V^{p,q}$ corresponds to the eigenspace for the character z^{p-q} . By abuse of notation, we still denote by $h: U(1) \rightarrow \text{Hg}(\mathbb{R})$ the restriction $h|_{U(1)}$. Note that $\mathbb{G}_m(\mathbb{R}) \cap U(1, \mathbb{R}) = \{\pm 1\}$ as subgroups of $\mathbb{S}(\mathbb{R}) = \mathbb{C}^*$. Thus, we have the following diagram:

$$(2.16) \quad \begin{array}{ccccc} h: \mathbb{C}^* & \longrightarrow & \text{Hg}(\mathbb{C}) & \longrightarrow & \text{GL}(V_{\mathbb{C}}) \\ \downarrow 2:1 & & \downarrow & & \\ \varphi: \mathbb{C}^* & \longrightarrow & \bar{G}(\mathbb{C}) & & \end{array}$$

Remark 2.17. Since the representation $V_{\mathbb{R}}$ is assumed irreducible, the lift to M (versus H) is equivalent to a Tate twist. However, with our definition of CY type (Definition 2.7), no Tate twist is allowed.

Modulo a factor of 2 (explained by (2.16)), the computation of the weights of h on $V_{\mathbb{C}}$ can be done at the level of Lie algebras. Since H is reductive and $\rho(G)$ is the derived subgroup, we have

$$(2.18) \quad \mathfrak{hg} := \text{Lie}(\text{Hg}) \cong \mathfrak{g} \oplus \mathfrak{a},$$

for some abelian Lie algebra \mathfrak{a} . Recall that a real irreducible representation $V_{\mathbb{R}}$ is of three possible types: *real type*, *complex type*, or *quaternionic type* depending on the behavior of the complexification:

$$(2.19) \quad V_{\mathbb{C}} = \begin{cases} V_+ & \text{real type} \\ V_+ \oplus V_-, V_+ \not\cong V_- & \text{complex type} \\ V_+ \oplus V_-, V_+ \cong V_- & \text{quaternionic type} \end{cases},$$

where V_{\pm} are irreducible $G(\mathbb{C})$ -representations and V_{\pm} are dual representation (i.e. $V_+^{\vee} \cong V_-$). Since

$$(2.20) \quad \text{End}_{\mathfrak{g}}(V_{\mathbb{R}}) = \begin{cases} \mathbb{R} & \text{real type} \\ \mathbb{C} & \text{complex type} \\ \mathbb{H} & \text{quaternionic type} \end{cases}$$

(e.g. [GGK11, p. 70]), we conclude that the abelian part \mathfrak{a} is either trivial, or possibly one dimensional in the complex or quaternionic case. Furthermore, in the complex or quaternionic case, when $V_{\mathbb{C}} = V_+ \oplus V_-$, a generator of the one-dimensional \mathfrak{a} corresponds to the obvious \mathbb{C}^* -action: $t \in \mathbb{C}^*$ acts as scaling by t on V_+ and scaling by t^{-1} on V_- .

In our situation, it is important to note the following property for the minuscule representation associated to the Hermitian symmetric domain \mathcal{D} .

Lemma 2.21. *Let $\mathcal{D} = G(\mathbb{R})/K$ be a Hermitian symmetric domain of non-compact type, and ϖ_i the associated minuscule weight. As a $G(\mathbb{R})$ -representation the minuscule representation V_{ϖ_i} is of real type for the following cases (see also table 1 for labeling): $\text{I}_{n,n} (A_{2n-1}, \alpha_n)$, $\text{IV}_{2n-1} (B_n, \alpha_1)$, $\text{III}_n (C_n, \alpha_n)$, $\text{IV}_{2n} (D_n, \alpha_1)$, $\text{II}_{2n} (D_{2n}, \alpha_{2n})$, $\text{EVII} (E_7, \alpha_7)$. Otherwise, V_{ϖ_i} is of complex type.*

Proof. If V_{ϖ_i} is not self-dual, then the representation is of complex type. The dual representation has highest weight $\tau\varpi_i$, where τ is the opposition involution. The condition $\tau\varpi_i = \varpi_i$ holds exactly for the cases listed above. To complete the proof, if $\tau\varpi_i = \varpi_i$, one needs to decide if the representation is real or quaternionic.

In our situation, it is standard that all these cases are real (e.g. [Gro94, §2]). Alternatively, one easily checks the reality of the representation using the criterion given by [GGK11, Theorem IV.E.4]. \square

Remark 2.22. The Hermitian symmetric domains for which the fundamental representation V_{ϖ_i} is of real type are precisely the domains with a tube domain realization (compare [Gro94, §1]). We will refer to these cases as *tube type* or *real type*, while the others will be referred as *non-tube* or *complex* cases.

Recalling that the projection of the differential of h on the \mathfrak{g} factor (see (2.18)) is H_0 , we conclude that in the real type case, the weights of h on $V_{\mathbb{C}}$ are:

$$(2.23) \quad \{2\varpi(H_0) \mid \varpi \in \mathcal{X}(V_+)\},$$

where $\mathcal{X}(V_+)$ denotes the weights of the irreducible $G(\mathbb{C})$ -representation V_+ (N.B. $V_+ \cong V_{\mathbb{C}}$ in this case). In the complex or quaternionic case, if \mathfrak{a} is 1-dimensional, the weights can all be shifted by a constant c . Since the weights of V_+ and V_- are opposite and the shifts act in opposite directions, we conclude that in the complex and quaternionic cases the weights of h on $V_{\mathbb{C}}$ are:

$$(2.24) \quad \{\pm 2(\varpi(H_0) - c) \mid \varpi \in \mathcal{X}(V_+)\}$$

for some constant c . In this situation, c and $\varpi(H_0)$ can (and typically are) rational numbers (with denominator dividing $2d$, where d is the connection index of the root system R). At the same time, the difference $\varpi(H_0) - c$ is a half-integer. This is explained by the fact that one might need to pass to a finite cover to lift h to

$$\tilde{h}: \mathbb{C}^* \rightarrow G(\mathbb{C}) \times T(\mathbb{C}) \xrightarrow{(\rho, \chi)} \mathrm{GL}(V_{\mathbb{C}})$$

so that it fits in the diagram (extending (2.16)):

$$(2.25) \quad \begin{array}{ccccc} & & \mathbb{C}^* & \longrightarrow & G(\mathbb{C}) \times \mathbb{C}^* & \xrightarrow{(\rho, \chi)} & \\ & \swarrow & \downarrow & & \downarrow & \searrow & \\ \mathbb{C}^* & \longrightarrow & \mathrm{Hg}(\mathbb{C}) & \longrightarrow & \mathrm{GL}(V_{\mathbb{C}}) & & \\ & \swarrow & \downarrow & & \downarrow & \nearrow & \\ & & \mathbb{C}^* & \longrightarrow & G(\mathbb{C}) & \xrightarrow{\rho} & \\ 2:1 \downarrow & & \downarrow & & \downarrow & & \\ & & \mathbb{C}^* & \longrightarrow & \bar{G}(\mathbb{C}) & & \end{array}$$

Convention 2.26. For a rational number p/q , the notations $W \left\langle \frac{p}{q} \right\rangle$ and $W \left| \frac{p}{q} \right\rangle$ will have the same meaning as in (2.9), but should be understood in the context of the above diagram. Namely, \mathbb{C}^* is a 1-PS of $G(\mathbb{C}) \times_{\mu_q} \mathbb{C}^*$, and the shift is by the character $\chi(t) = t^{-p}$ as in the diagram.

Lemma 2.27. *With notation as above, let λ be the highest weight of an irreducible factor V_+ of V . Possibly after replacing V_+ with V_- , we can assume that $\tau\lambda(H_0) \leq \lambda(H_0)$. Then a necessary condition that the irreducible representation $\rho: G \rightarrow \mathrm{GL}(V)$ arises from a Hermitian VHS of CY type over \mathcal{D} is*

$$(2.28) \quad \varpi(H_0) < \lambda(H_0) \text{ for all weights } \varpi \neq \lambda \text{ of } V_+.$$

Furthermore, the condition (2.28) is equivalent to the condition that λ is a multiple of the fundamental weight ϖ_i associated to the domain \mathcal{D} . In particular, as a $G(\mathbb{R})$ -representation V is either real or complex depending on \mathcal{D} is of tube type or not, and the quaternionic case does not arise.

Proof. Since all the weights of V_+ are obtained from λ by subtracting positive roots, it follows

$$\max_{\varpi \in \mathcal{X}(V_+)} \varpi(H_0) = \lambda(H_0).$$

Then, using the description of the weights of h on $V_{\mathbb{C}}$ (cf. (2.23) and (2.24)), we see that $\dim_{\mathbb{C}} V^{n,0} = 1$ implies that the above maximum is attained only for the weight λ (i.e. (2.28)). By applying the reflection in the root α_j , and using (2.14), we get

$$s_{\alpha_j}(\lambda)(H_0) = (\lambda - \lambda(\check{\alpha}_j) \cdot \alpha_j)(H_0) = \lambda(H_0),$$

(where $\check{\alpha}_j$ is the corresponding coroot). Since $s_{\alpha_j}(\lambda) \in \mathcal{X}(V_+)$, from (2.28), we conclude that $s_{\alpha_j}(\lambda) = \lambda$ for all $j \neq i$, i.e. $\lambda(\check{\alpha}_j)$ for all $j \neq i$, which is equivalent to $\lambda = n\varpi_i$. The last assertion follows Lemma 2.21. \square

Proof of Theorem 2.11. First consider the tube domain case. By Lemma 2.27, if V is a Hermitian VHS of CY type, then there is an irreducible summand V' (over \mathbb{C}) of V occurring in V with highest weight $n\varpi_i$. But then V' is a real representation, so that $V' = V$. The fact that V actually arises follows from Gross [Gro94]. More precisely, for $n = 1$, the construction of the Hermitian VHS \mathcal{V} over the tube domain \mathcal{D} is the content of [Gro94]. For $n > 1$, the (not in general irreducible) VHS $\text{Sym}^n \mathcal{V}$ is of CY type, and an irreducible summand will have highest weight $n\varpi_i$ and will remain of CY type. Finally, we have already noted that, in the real case, the Hodge group has to coincide with G (see (2.20)). Thus, once the representation ρ is fixed, the only possible variation in lifting φ to h is a Tate twist, which in turn is not allowed by Definition 2.7 (see also Remark 2.17). This completes the proof in the tube domain case.

In the non-tube domain case, a Hermitian VHS of CY type over \mathcal{D} gives a complex representation V with $V_{\mathbb{C}} = V_+ \oplus V_-$. After reordering, we may assume that the highest weight for V_+ is equal to $n\varpi_i$. Note that V_+ is not real. Let W be the representation of G corresponding to $n = 1$. Sheng-Zuo [SZ10] have noted that W carries a \mathbb{C} -Hodge structure, and hence the vector space $W \oplus W^{\vee}$ will carry an \mathbb{R} -Hodge structure. Typically, this structure will not be of CY type. From the description of the weights (2.24), it is immediate to see that after an iterated half-twist, one can arrange $V_+ \langle c \rangle \oplus V_- \langle c \rangle$ to be of CY type for all $n > 0$. Specifically, for $\lambda = n\varpi_i$,

$$(2.29) \quad c = \frac{1}{2} (\lambda(H_0) - \tau \lambda(H_0)) - \frac{1}{2}$$

will give a CY Hodge structure of minimal weight (compare §2.2). Any additional $-\frac{1}{2}$ half-twists will preserve the CY condition, but will increase the weight by 1. The additional half-twists are explained by the fact that the (generic) Hodge group is $\text{Hg}(\mathbb{C}) = G(\mathbb{C}) \cdot \mathbb{C}^*$ (in contrast to the real case, where $\text{Hg} = G$). The minimal weight satisfying the CY condition is equal to the real rank of Hg , and thus is one larger than the real rank of G . Finally, it is easy to see that there always exists a compatible polarization (cf. [GGK11, §IV.A (Step 4), p. 72]): in the complex case there exists an invariant Hermitian form on V_+ , which gives both an alternating and a symmetric form on V (the correct choice depending on the half-twist, or equivalently the weight). \square

Remark 2.30. There is a similar argument in Deligne [Del79, §1.3, especially Lemma 1.3.5]. Related computations, but with a different focus, occur in [Moo99, §3] (based on Zarhin [Zar84]) and [GGK11, Chapter IV].

2.2. List of Hodge representations of CY type. Here we discuss in more detail the canonical Hermitian VHS of CY type given by Theorem 2.11. Let \mathcal{D} is an irreducible Hermitian symmetric domain corresponding to (R, α_i) , where R is a root system and α_i is a special root (see Table 1 for a list of possibilities). Let V be the representation giving the VHS of CY type, a minuscule representation. Let $\tau = -w_0$ denote the opposition involution. As before, we distinguish two cases: real (or tube) case if $\tau\alpha_i = \alpha_i$, and complex otherwise. We denote by V_+ the irreducible summand of the $G(\mathbb{C})$ -representation $V_{\mathbb{C}}$ of highest weight ϖ_i . In the complex case, $V_{\mathbb{C}} = V_+ \oplus V_-$ and V_- has highest weight $\tau\varpi_i$ (another minuscule weight).

Label	(R, α_i)	G	K	\mathbb{R} -rank
$I_{p,q}$	(A_{p+q-1}, α_p)	$SU(p, q)$	$S(U(p) \times U(q))$	$\min(p, q)$
II_n	(D_n, α_n)	$SO^*(2n)$	$U(n)$	$\frac{n}{2}$
III_n	(C_n, α_n)	$Sp(n, \mathbb{R})$	$U(n)$	n
IV_{2n-1}	(B_n, α_1)	$Spin(2, 2n-1)$	$Spin(2) \times_{\mu_2} Spin(2n-1)$	2
IV_{2n}	(D_{n+1}, α_1)	$Spin(2, 2n)$	$Spin(2) \times_{\mu_2} Spin(2n)$	2
EIII	(E_6, α_1)	$E_{6,2}$	$U(1) \times_{\mu_4} Spin(10)$	2
EVII	(E_7, α_7)	$E_{7,3}$	$U(1) \times_{\mu_3} E_6$	3

TABLE 1. Hermitian symmetric domains of non-compact type

The set of roots Δ of R are partitioned into compact roots Δ_c and non-compact roots Δ_{nc} . If, as before (see §2.1), $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition and \mathfrak{h} is a compact Cartan subalgebra, we have

$$\begin{aligned} \mathfrak{k}_{\mathbb{C}} &= \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_c} \mathfrak{g}_{\alpha} = \mathbb{C} \cdot H_0 \oplus \mathfrak{g}'_{\mathbb{C}}, \\ \mathfrak{p}_{\mathbb{C}} &= \bigoplus_{\alpha \in \Delta_{nc}} \mathfrak{g}_{\alpha}, \end{aligned}$$

where \mathfrak{g}' denotes the semi-simple part of the Lie algebra \mathfrak{k} . The Dynkin diagram of the root system R' associated to $\mathfrak{g}'_{\mathbb{C}}$ is obtained from that of R by erasing the node corresponding to α_i . Note that the Weyl group $W(R)$ acts transitively on the weights $\mathcal{X}(V_+)$ of the minuscule representation V_+ , while the subgroup $W(R') \subset W(R)$ is the stabilizer of ϖ_i . Moreover, each weight subspace $V_{\varpi} \subset V_+$ for $\varpi \in \mathcal{X}(V_+)$ is 1-dimensional. We are interested in the weight decomposition of V_+ with respect to the cocharacter φ (corresponding to H_0). Using (2.14), one immediately sees (compare also Lemma 2.27, and [Moo99, §3]):

- (1) The weights of φ on W are: $\varpi_i(H_0), \varpi_i(H_0) - 1, \dots, w_0\varpi_i(H_0)$. Thus, the minimal weight of a Hodge structure on V will be $k := \varpi_i(H_0) + \tau(\varpi_i)(H_0) \in \mathbb{Z}_+$. Note that k coincides with the real rank of $G(\mathbb{R})$ and can be computed as the sum of the coefficients of the roots α_i and $\tau\alpha_i$ in the weight ϖ_i . For example, for (E_7, α_7) : $k = 2 \cdot \frac{3}{2} = 3$, and for (E_6, α_1) : $k = \frac{4}{3} + \frac{2}{3} = 2$.
- (2) The weight space corresponding to the weight $\varpi_i(H_0)$ is 1-dimensional (the CY condition).

- (3) The weight space corresponding to the weight $\varpi_i(H_0) - 1$ (corresponding to $V_+^{k-1,1}$) has dimension equal to half the number of non-compact roots (i.e. $\frac{1}{2}(|\Delta_R| - |\Delta_{R'}|)$). For example, for E_7 , we get $h^{2,1} = \frac{1}{2}(126 - 72) = 27$.

The real case (or equivalently \mathcal{D} is of tube domain type) is discussed in detail by Gross [Gro94]. For completeness, we list the cases with some relevant information:

2.2.1. $I_{n,n} (A_{2n-1}, \alpha_n)$: weight $2\varpi_n(H_0) = n$; $V = \bigwedge^n S$, where S is the standard representation; for $n = 3$, $h^{2,1} = 9$.

2.2.2. $IV_{2n-1} (B_n, \alpha_1)$: weight 2; V is the standard representation; this is the classical case of $K3$ type.

2.2.3. $III_n (C_n, \alpha_n)$: weight n ; V is the an irreducible factor of $\bigwedge^n S$, where S is the standard representation; for $n = 3$, $h^{2,1} = 6$; this case corresponds geometrically to the middle cohomology of an abelian n -fold.

2.2.4. $IV_{2n-2} (D_n, \alpha_1)$: weight 2; V is the standard representation; this is the classical case of $K3$ type.

2.2.5. $II_{2n} (D_{2n}, \alpha_{2n})$: weight n ; V is a half-spin representation; for $n = 3$, $h^{2,1} = 15$.

2.2.6. $EVII (E_7, \alpha_7)$: weight 3; V is the minuscule representation; $h^{2,1} = 27$.

The remaining cases are of complex type. The \mathbb{C} -Hodge structure induced by (a lift of) φ on V_+ was studied by Sheng–Zuo [SZ10]. We are interested in an \mathbb{R} -Hodge structure on V of CY type. As explained, this can be obtained by considering $V_+ \langle c \rangle \oplus V_- \langle c \rangle$ for an appropriate shift c . It is interesting to note that minimal weight Hodge structure associated to the representation V is not of CY type. The CY Hodge structure on V is obtained by applying a half-twist to this minimal weight Hodge structure. The relevant cases are:

2.2.7. $I_{p,q} (A_{p+q-1}, \alpha_p)$ for $p < q$: Note first $\tau\alpha_p = \alpha_q$, $P/Q \cong \mathbb{Z}/(p+q)$, where Q and P are the root and weight lattices respectively. We have

$$\begin{aligned}\varpi_p(H_0) &= \frac{pq}{p+q}, \\ \tau(\varpi_p)(H_0) &= \frac{p^2}{p+q},\end{aligned}$$

and thus the minimal weight for a Hodge structure on V will be $k = \varpi_p(H_0) + \tau(\varpi_p)(H_0) = p$. To obtain a CY Hodge structure, the minimal weight will be $p+1$. Specifically, the weights of the cocharacter $\varphi_{\mathbb{C}}$ on V_+ are:

$$\frac{pq}{(p+q)}, \frac{pq}{(p+q)} - 1, \dots, -\frac{p^2}{p+q}$$

while those on the dual representation V_- are:

$$\frac{p^2}{(p+q)}, \frac{p^2}{(p+q)} - 1, \dots, -\frac{pq}{p+q}.$$

To obtain a (minimal weight) Hodge structure, one needs to shift the weights so that $\frac{pq}{(p+q)} - c = \frac{p^2}{(p+q)} + c$ (recall that the Hodge group $\text{Hg}(\mathbb{C}) = G(\mathbb{C}) \cdot \mathbb{C}^*$ in the complex case; see Convention 2.26 for the meaning of the fractional shift; also

compare to (2.29)). We get that $V_+ \left\langle \frac{p(q-p)}{2(p+q)} \right| \oplus V_- \left| \frac{p(q-p)}{2(p+q)} \right\rangle$ will carry a Hodge structure of weight p . By applying a half-shift, we obtain a CY Hodge structure of weight $p+1$:

$$(2.31) \quad V_+ \left\langle \frac{p(q-p)}{2(p+q)} - \frac{1}{2} \right| \oplus V_- \left| \frac{p(q-p)}{2(p+q)} - \frac{1}{2} \right\rangle.$$

The Hodge numbers are easily computed by noting that $V_+ = \bigwedge^p S$, where S is the standard representation of $\mathrm{SU}(p, q)$.

We are particularly interested in the weight 3 CY case. Since we need $p+1 \leq 3$, we get $p=1$ or $p=2$. If $p=1$, the associated domain $I_{1,q}$ is the q -dimensional complex ball. The minimal weight for a Hermitian VHS is 1 in this case, and the minimal weight for a Hermitian VHS of CY type is 2 ($K3$ type). To obtain a weight 3 VHS, we need to apply an additional half-twist. It is immediate to see that $h^{2,1} = q$. This case is somewhat special among the complex cases, in the sense that $V_+ = V^{3,0} \oplus V^{2,1}$ and $V_- = V^{1,2} \oplus V^{0,3}$, i.e. there is no “mixing” of V_+ and V_- in $V^{2,1}$. Also note that the VHS of weight 3 is of maximal dimension, i.e. $\dim \mathcal{D} = h^{2,1}$. If $p=2$, the minimal weight for a Hermitian VHS of CY type over $I_{2,q}$ is $3 = p+1$. Since $V_+ = \bigwedge^2 S$, we get the dimensions of the weight spaces for V_+ (w.r.t. φ) to be $1, q+1, \frac{(q+2)(q-1)}{2}$. Thus,

$$h^{2,1} = h_+^{2,1} + h_-^{2,1} = (q+1) + \frac{(q+2)(q-1)}{2} = \frac{q(q+3)}{2}$$

for the resulting weight 3 Hodge structure (where $h_{\pm}^{2,1}$ denotes the dimension of $(2,1)$ space on V_{\pm} after the shift). On the other hand, the dimension of $I_{2,q}$ is $2q$, showing that the Hermitian variation of Calabi–Yau type on $I_{2,q}$ is not maximal for $q > 1$.

Remark 2.32. It is interesting to note that the case $I_{1,n}$ occurs quite often in the geometric context. For example, the embedding of $I_{1,n}$ in a Siegel space, corresponding to the minimal weight Hermitian VHS on $I_{1,n}$, occurs in Deligne–Mostow [DM86] uniformization of the moduli of points in \mathbb{P}^1 by complex balls. Kondo has realized most of these examples via embeddings of $I_{1,n}$ into Type IV domains by using $K3$ surfaces, corresponding to the minimal weight Hermitian VHS on $I_{1,n}$ of Calabi–Yau type (see [DK07] for this and related examples). Finally, Rohde [Roh09] and van Geemen [GvG10] have produced VHS over $I_{1,n}$ associated to families of Calabi–Yau 3-folds. The half-twist construction explains (and was motivated by) these examples (see [vG01, vGI02]).

2.2.8. $(D_{2n-1}, \alpha_{2n-1})$: $\tau\alpha_{2n-1} = \alpha_{2n-2}$, $P/Q \cong \mathbb{Z}/4$. We have

$$\begin{aligned} \varpi_{2n-1}(H_0) &= \frac{2n-1}{4}, \\ \tau(\varpi_{2n-1})(H_0) &= \frac{2n-3}{4}. \end{aligned}$$

Thus the weights on V_+ are: $\frac{2n-1}{4}, \frac{2n-1}{4} - 1, \dots, -\frac{2n-3}{4}$. The minimal twist that gives a CY Hodge structure is

$$V_+ \left\langle -\frac{1}{4} \right| \oplus V_- \left| -\frac{1}{4} \right\rangle$$

and it has weight n . The representation V_+ is a half-spin representation. It follows, for $n=3$, $h^{2,1} = 2^{2n-2} - 1 = 15$.

2.2.9. (E_6, α_1) : $\tau\alpha_1 = \alpha_6$, $P/Q \cong \mathbb{Z}/3$. We have

$$\begin{aligned}\varpi_1(H_0) &= \frac{4}{3}, \\ \tau\varpi_1(H_0) &= \frac{2}{3}.\end{aligned}$$

The minimal twist that gives a CY Hodge structure is

$$V_+ \left\langle -\frac{1}{6} \right| \oplus V_- \left| -\frac{1}{6} \right\rangle$$

of weight 3. More explicitly, consider the direct sum $V_+ \oplus V_-$ of the two minuscule representations of E_6 . There is a lift $\tilde{\varphi}$ of the cocharacter φ of \bar{G} to the simply connected form G so that it acts with weights (and dimensions of weight spaces):

weight	8	4	2	-2	-4	-8
V_+	1		16		10	
V_-		10		16		1
$V_{\mathbb{C}}$	1	10	16	16	10	1

(N.B. the minimal lift of φ to G would act with weights 4, 2, 1, -1, -2, -4. Due to the diagram (2.16), we need to take a double cover of it). We then consider the 1-parameter subgroup given by

$$\begin{aligned}\mathbb{C}^* &\rightarrow G(\mathbb{C}) \times \mathbb{C}^* \\ t &\rightarrow (\tilde{\varphi}(t), t)\end{aligned}$$

where $t \in \mathbb{C}^*$ acts on $V_+ \oplus V_-$ by character t on V_+ and t^{-1} on V_- . The twisted weights (and dimensions of weight spaces) will be

weight	3	1	-1	-3
$V_{\mathbb{C}}$	1	26	26	1

i.e. we obtain a CY Hodge structure of weight 3 with $h^{2,1} = 26$. As in the other complex cases, the minimal weight Hodge structure of E_6 type will be of weight 2 = rank $_{\mathbb{R}}$ EIII with Hodge numbers 11, 32, 11, which is no longer of CY type. Also note that $K = \mathrm{U}(1) \times_{\mu_4} \mathrm{Spin}(10)$. Then the dimensions of the $V^{p,q}_{\pm}$ (and $V^{p,q}_{\pm}$) spaces can be also computed by decomposing V_+ as a $\mathrm{Spin}(10)$ -representation. Explicitly, we have

$$V_+ \cong \mathbb{C} \oplus W_{\varpi_1} \oplus W_{\varpi_5}$$

as a D_5 -representation (see also [SZ10, §4.5]). Here the first summand is the standard representation, while the second is the half-spin representation of $\mathrm{Spin}^*(10)$.

Notation 2.33. We encode a Hermitian VHS of CY type with the notation: $(R, \alpha_i; \lambda)\{c\}$, where (R, α_i) determines the domain \mathcal{D} , λ is the highest weight of the irreducible complex representation V_+ , and, in the complex case, c denotes a half-twist (see (2.8)). The possible fractional meaning of c is explained by Convention 2.26.

Summarizing the above discussion, we obtain the following list of Hermitian VHS of CY type for weight 3.

Corollary 2.34. *The Hermitian VHS of CY type for weight 3 are:*

- i) *Four primitive real cases: $\mathrm{I}_{3,3}$ ($A_5, \alpha_3; \varpi_3$), III_3 ($C_3, \alpha_3; \varpi_3$), II_6 ($D_6, \alpha_6; \varpi_6$), and EVII ($E_7, \alpha_7; \varpi_7$), corresponding to the weight 3 cases in Gross [Gro94].*

- ii) *Re-embeddings of lower weight cases:* $\mathfrak{H} (A_1, \alpha_1; 3\varpi_1)$.
- iii) *Complex cases: two infinite series:* $I_{1,n} (A_n, \alpha_1; \varpi_1) \left\{ -\frac{n+3}{2(n+1)} \right\}$ and $I_{2,n} (A_{n+1}, \alpha_2; \varpi_2) \left\{ \frac{n-6}{2(n+2)} \right\}$, and two isolated case: $II_5 (D_5, \alpha_5; \varpi_5) \left\{ -\frac{1}{4} \right\}$, and $EIII (E_6, \alpha_1; \varpi_1) \left\{ -\frac{1}{6} \right\}$.
- iv) *Reducible cases:* $\mathfrak{H} \times IV_n$ (in particular $\mathfrak{H} \times \mathfrak{H} \times \mathfrak{H}$) or $\mathfrak{H} \times I_{1,n}$.

Remark 2.35. Note that the cases that give maximal horizontal subvarieties (i.e. $h^{2,1} = \dim \mathcal{D}$) are those of (i), (ii), (iii) for $I_{1,n}$, and (iv) for $\mathfrak{H} \times \mathcal{D}_n$, i.e. the real cases and the complex unit ball case. Most of the remaining cases (all of complex type) can be embedded into maximal weight three Hermitian VHS over some other Hermitian domain (of real type) and be understood as Noether–Lefschetz subloci with weak complex multiplication. Specifically, if \mathcal{D} carries a VHS of CY type, and $\mathcal{D}' \hookrightarrow \mathcal{D}$ is a totally geodesic embedding, then, by restriction, \mathcal{D}' also carries a VHS of CY type. Satake [Sat65] and Ihara [Iha67] have classified all the holomorphic, totally geodesic embeddings $\mathcal{D}' \hookrightarrow \mathcal{D}$ among Hermitian symmetric domains. Applying this classification to our situation, we obtain the following embeddings of the complex cases into maximal VHS:

- a) $I_{2,6} \hookrightarrow EVII$;
- b) $II_5 \hookrightarrow II_6$;
- c) $EIII \hookrightarrow EVII$;
- d) $\mathfrak{H} \times I_{1,n} \hookrightarrow \mathfrak{H} \times IV_{2n}$ (induced from $I_{1,n} \hookrightarrow IV_{2n}$).

The case $I_{2,n}$ ($n > 6$) does not embed in a maximal weight three Hermitian VHS of CY type.

3. ENDOMORPHISMS OF CY HODGE STRUCTURES

Theorem 2.11 is a classification of Hermitian VHS of CY type over \mathbb{R} , or over \mathbb{Q} under the additional assumption 2.5 that the irreducible representation V of G is defined over \mathbb{Q} and remains irreducible over \mathbb{R} . To understand the situation in general, and to put the real and complex cases in the proper context, we study the endomorphisms of a Hodge structure of CY type. This section is inspired by the work of Zarhin [Zar83] on Hodge groups for $K3$ surfaces.

We recall that for a \mathbb{Q} -Hodge structure (V, h) , the endomorphism algebra is defined as

$$E := \text{End}_{\text{Hg}}(V) = \{f: V \rightarrow V \mid f \text{ is } \mathbb{Q}\text{-linear map s.t. } f_{\mathbb{C}}(V^{p,q}) \subseteq V^{p,q}\}.$$

Under the assumption that V is a simple Hodge structure (i.e. it contains no non-trivial sub-Hodge structures), E is a finite dimensional division algebra over \mathbb{Q} . If V is of CY type, by an argument essentially due to Zarhin [Zar83, Theorem 1.6(a), Theorem 1.5], we obtain:

Theorem 3.1. *Let (V, h) be a simple \mathbb{Q} -Hodge structure of CY type. Then the endomorphism algebra E is a number field. If additionally (V, h) is polarizable, then either*

- i) *the real case: E is a totally real number field, or*
- ii) *the complex case: E is a CM field (i.e. a purely imaginary extension of a totally real number field).*

Proof. Since $V^{n,0}$ is preserved by a Hodge endomorphism, we get a non-trivial morphism of algebras:

$$(3.2) \quad \begin{aligned} \epsilon: \operatorname{End}_{\operatorname{Hg}}(V) &\rightarrow \operatorname{End}_{\mathbb{C}}(V^{n,0}) \\ f &\mapsto (f_{\mathbb{C}})|_{V^{n,0}}. \end{aligned}$$

Since $V^{n,0}$ is 1-dimensional and E is a division algebra, we conclude $\operatorname{End}_{\mathbb{C}}(V^{n,0}) \cong \mathbb{C}$ and then that $E \subset \mathbb{C}$ (via ϵ) and thus it is a number field.

A polarization Q , defines an involution † , the *Rosati involution*, on E by

$$Q(fv, w) = Q(v, f^\dagger w), \text{ for all } v, w \in V, f \in E$$

(e.g. [Moo99, (1.20)]). The proposition follows by restricting Albert's classification of involutive division algebras (e.g. [Moo99, (1.19)]) to the case of number fields. \square

From the perspective of endomorphisms, the real and complex cases can be understood as follows:

Corollary 3.3. *Let (V, h) be a polarizable \mathbb{Q} -Hodge structure of CY type. Let M be the Mumford–Tate group and $G = M_{\operatorname{der}}$ the derived subgroup. Assume 2.5 holds (i.e. $V_{\mathbb{R}}$ is irreducible and $G(\mathbb{R})$ is simple), then either*

- i) *the real case: $E = \mathbb{Q}$ and $V_{\mathbb{R}}$ is a $G(\mathbb{R})$ -representation of real type, or*
- ii) *the complex case: $E = \mathbb{Q}[\sqrt{-d}]$ for some positive square-free integer d and $V_{\mathbb{C}}$ is a $G(\mathbb{R})$ -representation of complex type.*

Conversely, let V , G , and E be as above. Assume V is an irreducible $G(\mathbb{Q})$ -representation and E is either \mathbb{Q} or $\mathbb{Q}[\sqrt{-d}]$. Then $V_{\mathbb{R}}$ is an irreducible $G(\mathbb{R})$ -representation.

Proof. It is well known that E^* is the group of \mathbb{Q} -points of the centralizer in $\operatorname{GL}(V)$ of the Mumford–Tate group M (e.g. [Moo99, (1.23)]). Thus, $E^*(\mathbb{R})$ and $G(\mathbb{R})$ are commuting subgroups of $\operatorname{GL}(V_{\mathbb{R}})$. Also,

$$\mathbb{G}_m(\mathbb{Q}) \subseteq Z(M) \subseteq T_F := \operatorname{Res}_{F/\mathbb{Q}}(\mathbb{G}_{m,F}),$$

where F is the center of E ($F = E$ here). Additionally, in the polarized case:

$$(3.4) \quad Z(\operatorname{Hg}) \subseteq \{x \in T_F \mid x \cdot \bar{x} = 1\}.$$

where $\operatorname{Hg} \subset M$ is the Hodge group (e.g. [Moo99, (1.23)]).

Clearly, $\operatorname{End}_{\mathfrak{m}_{\mathbb{R}}}(V_{\mathbb{R}}) \subseteq \operatorname{End}_{\mathfrak{g}_{\mathbb{R}}}(V_{\mathbb{R}})$, where \mathfrak{m} is the Lie algebra of M . Since $V_{\mathbb{R}}$ is irreducible and by (2.20), we obtain that $E(\mathbb{R}) = E \otimes_{\mathbb{Q}} \mathbb{R}$ is either \mathbb{R} or \mathbb{C} . This gives that $E = \mathbb{Q}$ or $\mathbb{Q}[\sqrt{-d}]$. In particular, if $V_{\mathbb{R}}$ is a real representation, then $E = \mathbb{Q}$. Conversely, if $E = \mathbb{Q}$, it follows that G coincides with the Hodge group (see (3.4), also compare [Moo99, (1.24)]), but then the centralizer of G is $E^* = \mathbb{Q}^*$. We claim that this implies that the representation V is absolutely irreducible (and in particular it is if real type). For, if not, we have a decomposition $V_{\mathbb{C}} = V_1 \oplus \cdots \oplus V_n$ as a $G(\mathbb{C})$ -representation. Then $G(\mathbb{C})$ commutes with a n -dimensional torus $(\mathbb{C}^*)^n$, which is a contradiction (to the fact that the centralizer of $G = \operatorname{Hg}$ is the group of scalars \mathbb{Q}^*). The case $E = \mathbb{Q}[\sqrt{-d}]$ is similar. \square

Remark 3.5. We recall that a Hodge structure (V, h) is said to have *weak real multiplication* or *weak complex multiplication (CM)* by F if there is an inclusion $F \hookrightarrow E = \operatorname{End}_{\operatorname{Hg}}(V)$ for a totally real field or respectively CM field F (see [GGK11, Definition V.B.1(i)]). If V has weak CM, then one can apply the half-twist construction (see [vG01]). The previous corollary gives a more intrinsic description of

the distinction between the real and complex case and explains the occurrence of half-twists in Theorem 2.11.

3.1. The general case. We now consider the case that the endomorphism algebra is not \mathbb{Q} or $\mathbb{Q}[\sqrt{-d}]$. To fix the notation, let (V, h, Q) be a weight k polarized Hodge structure of CY type. Let E be the field of endomorphisms (cf. Theorem 3.1). We denote by $\bar{}$ the complex conjugation on E and by $E_0 = \{x \in E \mid \bar{x} = x\}$ the totally real sub-field of E . Let $d = [E_0 : \mathbb{Q}]$ and $\{\sigma_0, \dots, \sigma_{d-1}\}$ be the embeddings of E_0 in \mathbb{R} . Note that E is endowed with a preferred embedding into \mathbb{C} , $\epsilon: E \hookrightarrow \mathbb{C}$, given by (3.2). The restriction $\epsilon|_{E_0}$ gives a preferred embedding of E_0 into \mathbb{R} , which we denote also by ϵ and we set $\sigma_0 = \epsilon$.

Note that V is naturally an E_0 -vector space (via $(f, v) \in \text{End}(V) \times V \mapsto f(v) \in V$) with $\dim_{\mathbb{Q}} V = d \dim_{E_0} V$. Furthermore,

$$(3.6) \quad V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R} = \bigoplus_{i=1}^d V \otimes_{E_0, \sigma_i} \mathbb{R}.$$

Denoting $V_i = V \otimes_{E_0, \sigma_i} \mathbb{R}$, we obtain

$$V_{\mathbb{R}} = \bigoplus_i V_i$$

a decomposition into \mathbb{R} -Hodge structures (defined over E_0). By construction, V_0 is weight n Hodge structure of CY type (the other components V_i have lower weight). The Hodge group Hg of the Hodge structure V is a reductive group defined over \mathbb{Q} . Similarly to (3.6), at the level of Lie algebras we have a decomposition:

$$\mathfrak{hg}_{\mathbb{R}} = \bigoplus_i \mathfrak{hg}_i$$

such that \mathfrak{hg}_i acts on V_i . Furthermore,

$$\text{End}_{\mathfrak{hg}_i}(V_i) = \begin{cases} \mathbb{R} & \text{real case} \\ \mathbb{C} & \text{complex case} \end{cases}$$

(cf. [Zar83, §1.9.4]).

The Hodge structures V_i and the Lie algebras \mathfrak{hg}_i are defined over E_0 . In particular, if Hg_0 is the group associated to \mathfrak{hg}_0 , we get a representation $\rho_0: \text{Hg}_0 \rightarrow \text{GL}(V_0)$ defined over E_0 . The Hodge structure on V_0 is determined then by a morphism of algebraic groups:

$$h': S^1 \rightarrow \text{Hg}_0(\mathbb{R}) \xrightarrow{\rho_0} \text{GL}(V_0).$$

On the other hand, recall that the Hodge structure on V corresponds to a morphism

$$h: S^1 \rightarrow \text{Hg}(\mathbb{R}) \rightarrow \text{GL}(V_{\mathbb{R}}).$$

Let $X_h \in \mathfrak{hg}_{\mathbb{R}} = \text{Lie}(\text{Hg})$ and $X_{h'} \in \mathfrak{hg}_0 = \text{Lie}(\text{Hg}_0)$ be the differentials of h and h' respectively. As in Zarhin [Zar83, Remark 1.9.3] (see also [Zar83, 0.3.3]), we note that $X_h \in \mathfrak{hg}_0 \subset \mathfrak{hg}_{\mathbb{R}}$ and in fact $X_h = X_{h'}$. It follows that

- i) the real Lie algebras \mathfrak{hg}_i are of compact type for $i \neq 0$;
- ii) the Hodge structure on V is determined from that on V_0 by twisting by the Galois action as explained in [Zar83, Remark 1.9.4] (i.e. the other components V_i and \mathfrak{hg}_i are obtained by twisting the action of \mathfrak{hg}_0 on V_0 by the Galois group $\text{Gal}(\tilde{E}_0/\mathbb{Q})$, where \tilde{E}_0 is a Galois extension $\mathbb{Q} \subset E_0 \subseteq \tilde{E}_0$).

In fact, note that viewing V_0 as a vector space defined over E_0 , we have $V = \text{Res}_{E_0/\mathbb{Q}}(V_0)$ as \mathbb{Q} -vector space and $\text{Hg} \subseteq \text{Res}_{E_0/\mathbb{Q}}(\text{Hg}_0)$.

Remark 3.7. Let F be a totally real field, and G an algebraic group over F . Then

$$\text{Res}_{F/\mathbb{Q}}(G)(\mathbb{R}) = \prod_{\sigma: F \rightarrow \mathbb{R}} G(\mathbb{R}_\sigma),$$

where $G(\mathbb{R}_\sigma)$ is the extension of scalars via the embedding $\sigma: F \rightarrow \mathbb{R}$ (see [Mil11a, §4.8 on p. 39]). The twisting by the Galois action is explained in [Mil11a, p.40].

We are interested in classifying Hermitian symmetric domains $\mathcal{D} = G(\mathbb{R})/K$ that parametrize VHS \mathcal{V} of CY type. Considering the Hodge structure (V, h) corresponding to a generic point of \mathcal{D} , and applying the framework described above, we get: The group $\text{Hg}_0 \subset \text{Hg}(\mathbb{R})$ (with notation as above) has the property that its derived subgroup $G(\mathbb{R})$ is of Hermitian type. Additionally, as explained above, the cocharacter $h: S^1 \rightarrow \text{Hg}(\mathbb{R}) \rightarrow \text{GL}(V_{\mathbb{R}})$ giving the Hodge structure factors through Hg_0 and then projects to a cocharacter φ for G of Hermitian type (as in §2.1). Thus, we are in the situation of Theorem 2.11. The only difference in the assumption (2.5) is that we assume that G (and Hg) and the representation $\rho: G \rightarrow \text{GL}(V)$ are defined over a totally real number field E_0 instead of over \mathbb{Q} . Conversely, to produce a Hermitian VHS of CY type over \mathcal{D} with generic endomorphism algebra E with associated totally real field E_0 , one should start with a fixed embedding $E_0 \subset \mathbb{R}$, an E_0 -form of G , a representation $\rho: G \rightarrow \text{GL}(V)$ defined over E_0 satisfying the assumption 2.5, but with \mathbb{Q} replaced by E_0 and $V_{\mathbb{R}}$ equal to $V \otimes_{E_0} \mathbb{R}$. Then a \mathbb{Q} -VHS is easily obtained by applying the $\text{Res}_{E_0/\mathbb{Q}}$ functor to V and G . In particular, $\text{Res}_{E_0/\mathbb{Q}}(G) \subseteq \text{Hg}$, where Hg is the generic Hodge group of the VHS over \mathcal{D} . Note that a necessary and sufficient condition for the embedding $E_0 \subset \mathbb{R}$ and the E_0 -form of G to give a Hermitian \mathbb{Q} -VHS is the condition (i) above, namely that the twisted forms of G have to be of compact type. This can be expressed in terms of the Galois theory of E_0/\mathbb{Q} .

3.2. An example: $K3$ type (cf. [Zar83], [vG08]). An example of the behavior described above is the case of Hermitian VHS of $K3$ type. Specifically, if the endomorphism algebra is a totally real field, the following holds (compare [vG08, Theorem 2.8], [Zar83, Theorem 2.2.1]):

Theorem 3.8 (Zarhin). *Let (V, h, Q) be a $K3$ -type Hodge structure with $E = \text{End}_{\text{Hg}}(V) = E_0$ a totally real field. Then, the Hodge group Hg satisfies:*

$$\text{Hg} = \text{Res}_{E_0/\mathbb{Q}}\text{SO}(V, \tilde{Q}), \quad \text{Hg}(\mathbb{R}) \cong \text{SO}(2, m-2) \times \text{SO}(m, \mathbb{R})^{d-1},$$

and then $\text{Hg}(\mathbb{C}) \cong \text{SO}(m, \mathbb{C})^d$, where \tilde{Q} is the unique symmetric E_0 -bilinear form on V such that $\text{Nm}(\tilde{Q}) = Q$. The representations of these Lie groups on the $d \cdot m$ -dimensional vector spaces $V_{\mathbb{R}}$ and $V_{\mathbb{C}}$ are the direct sum of the standard representations of the factors.

Furthermore, one can construct Hermitian VHS of $K3$ type with the prescribed weak real multiplication by any totally real field E_0 over a Type IV domain \mathcal{D} (associated with $\text{SO}(2, m-2)$). Essentially, after fixing an embedding $\epsilon: E_0 \hookrightarrow \mathbb{R}$, one considers V an E_0 -vector space of dimension m with an E_0 -symmetric bilinear form \tilde{Q} of signature $(2, m-2)$ and proceeds as usual. The result will be a VHS of

$K3$ type over \mathcal{D} a Type IV domain, with

$$\mathcal{D} = \left\{ \omega \in \mathbb{P}(V \otimes_{E_0, \epsilon} \mathbb{C}) \mid \tilde{Q}(\omega, \omega) = 0, \tilde{Q}(\omega, \bar{\omega}) > 0 \right\}.$$

The only difference to the usual situation (when V is defined over \mathbb{Q}) is that the Hodge structures involved are defined over E_0 . By twisting by the Galois action (e.g. as underlying vector space consider $\text{Res}_{E_0/\mathbb{Q}} V$), it is easy to produce a Hermitian VHS defined over \mathbb{Q} and parametrized by \mathcal{D} . The only subtle point in the construction is the choice of \tilde{Q} such that the twisted forms of \tilde{Q} are negative definite. This roughly boils down to the choice of two elements $a_1, a_2 \in E_0$ such that $\epsilon(a_i) > 0$ and $\sigma(a_i) < 0$ for all other real embeddings $\sigma: E_0 \hookrightarrow \mathbb{R}$. For further details see van Geemen [vG08] (esp. Lemma 3.2).

The case when the endomorphism algebra E is of CM type is similar. Namely, Zarhin [Zar83, Theorem 2.3.1] shows that the Hodge group is $\text{Hg} \cong \text{Res}_{E_0/\mathbb{Q}} \text{U}(V, \tilde{Q})$ for some E_0 -Hermitian form \tilde{Q} extending Q on V .

Remark 3.9. In general, for a Hodge structure (V, φ) of CY type, using the same arguments as in [Zar83], one can prove

$$\text{Hg} \subseteq \begin{cases} \text{Res}_{E_0/\mathbb{Q}} \text{Sp}_{E_0}(V, \tilde{Q}) & \text{if } E \text{ is totally real and odd weight} \\ \text{Res}_{E_0/\mathbb{Q}} \text{SO}_{E_0}(V, \tilde{Q}) & \text{if } E \text{ is totally real and even weight} \\ \text{Res}_{E_0/\mathbb{Q}} \text{U}_{E_0}(V, \tilde{Q}) & \text{if } E \text{ is a CM field} \end{cases}$$

(see also [Moo99, (1.22)]). In general, for weights higher than 2, the equality does not hold, as there might exist higher height Hodge tensors.

4. EXPLICIT REALIZATION OF HORIZONTAL SUBVARIETIES: REAL CASE

For the remainder of this paper, we shall only be concerned with the weight three CY case. In this and the next section, our goal will be to give a concrete description of a class of horizontal subvarieties \hat{Z} of $\check{\mathbf{D}}$. In case \hat{Z} is globally a closed subvariety of $\check{\mathbf{D}}$, $Z = \hat{Z} \cap \mathbf{D}$ will then be semi-algebraic in \mathbf{D} .

A local description of horizontal subvarieties $\hat{Z} \subset \check{\mathbf{D}}$ is well known (Bryant–Griffiths [BG83]) under a mild non-degeneracy condition satisfied in case Z is the image of the period map for a family of Calabi–Yau threefolds. Namely, if X is a Calabi–Yau threefold then the first order deformations of X are unobstructed, by the Tian–Todorov theorem, and the Gauss–Manin connection induces an isomorphism $H^1(X; T_X) \rightarrow \text{Hom}(H^0(X; \Omega_X^3), H^1(X; \Omega_X^2))$. Thus, roughly speaking, the period map is a local embedding of the moduli space, and the image subvariety Z satisfies: *For every choice of local coordinates z_1, \dots, z_h on Z , if $\omega(z)$ is a local section of the Hodge bundle F^3 , then the derivatives $\partial\omega/\partial z_i$ span F^2/F^3 .* We will assume in this section that Z satisfies this property and that there is a global choice of coordinates adapted to this situation, which we shall describe below. This global description is well-adapted to questions involving real or rational structures (for example, the existence of maximally unipotent monodromy) and is an analogue of the Cayley transform, which is an unbounded realization of a Hermitian symmetric space. In the next section, we shall describe the analogue of the Harish-Chandra embedding of a Hermitian symmetric space as a bounded symmetric domain.

4.1. Local description of the horizontal subvarieties.

Notation 4.1. The standard symplectic basis on a lattice Λ of rank $2h + 2$ will be written as $e_0, e_1, \dots, e_h, f_1, \dots, f_h, f_0$ with $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$ for all i, j and $\langle e_i, f_j \rangle = \delta_{ij}$. In particular, the symplectic lattice $(\mathbb{Z}^{2h+2}, \langle \cdot, \cdot \rangle)$ comes with a fixed filtration W_\bullet defined by

$$\begin{aligned} W_0 = W_1 = \mathbb{Z} \cdot e_0 &\subseteq W_2 = W_3 = \text{span}\{e_0, e_1, \dots, e_h\} \\ &\subseteq W_4 = W_5 = \text{span}\{e_0, e_1, \dots, e_h, f_1, \dots, f_h\} \subseteq W_6 = \mathbb{Z}^{2h+2}. \end{aligned}$$

Hence we can speak of a large radius limit, or equivalently of integral symplectic matrices T preserving the filtration (which will in general be maximally unipotent). We will also relax the condition that the e_i, f_i be integral and sometimes just assume that they are a fixed symplectic basis of a k -vector space V with a symplectic form defined over k .

Let \mathbf{D} be the classifying space of Hodge structures of weight 3 on $\Lambda_{\mathbb{C}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ satisfying $h^{3,0} = h^{0,3} = 1$, and hence $h^{2,1} = h^{1,2} = h$, and let $\check{\mathbf{D}}$ denote the compact dual of \mathbf{D} . We begin with a local description of a horizontal subvariety \hat{Z} of $\check{\mathbf{D}}$, essentially going back to Bryant–Griffiths [BG83] (see also [Fri91], [Voi99]). Here the Hodge–Riemann inequalities will be irrelevant.

Definition 4.2. Let V be a complex vector space endowed with a symplectic form $\langle \cdot, \cdot \rangle$. A *Legendrian immersion* from a manifold X to $\mathbb{P}V$ is an immersion $\iota: X \rightarrow \mathbb{P}V$ such that, for every $x \in X$, the subspace of V corresponding to $\iota_*(T_x X)$ is a Lagrangian subspace of V .

By definition a Legendrian immersion induces a filtration of V for each $x \in X$: given $x \in X$, set

$$F^3 = \mathbb{C} \cdot \tilde{x} \subseteq F^2 = \tilde{T} \subseteq F^1 = (F^3)^\perp \subseteq F^0 = V,$$

where $\tilde{x} \in \tilde{T}$ are affine lifts of $\iota(x)$ and $\iota_*(T_x X)$. If $\iota: X \rightarrow \mathbb{P}V$ is a Legendrian immersion, then its first prolongation is an immersion

$$\iota^{(1)}: X \rightarrow \check{\mathbf{D}}$$

whose image is (locally on X) an h -dimensional horizontal submanifold of $\check{\mathbf{D}}$. Conversely, if \hat{Z} is any h -dimensional horizontal submanifold of $\check{\mathbf{D}}$ satisfying the appropriate non-degeneracy condition and $\iota: \hat{Z} \rightarrow \mathbb{P}V$ corresponds to taking the complex line $F^3 \subseteq V$, then $\hat{Z} \rightarrow \check{\mathbf{D}}$ is the first prolongation of ι .

For an explicit construction, let $\omega(z)$ be a generator of F^3 , where $z = (z_1, \dots, z_h)$ is a set of local coordinates on \hat{Z} . We assume that ω is in the normalized form:

$$(4.3) \quad \omega(z) = \psi e_0 + \sum_{i=1}^h \alpha_i e_i + \sum_{i=1}^h z_i f_i + f_0.$$

Here the z_i are local coordinates on the subvariety \hat{Z} and ψ and α_i are functions of the z_i .

Proposition 4.4. *With the above normalization (4.3), the local description of a horizontal subvariety $Z = Z_\varphi$ is given by*

$$(4.5) \quad \omega = \left(\varphi - \frac{1}{2} \sum_{i=1}^h z_i \frac{\partial \varphi}{\partial z_i} \right) e_0 + \frac{1}{2} \sum_{i=1}^h \frac{\partial \varphi}{\partial z_i} e_i + \sum_{i=1}^h z_i f_i + f_0.$$

for φ a holomorphic function of the z_i , i.e. $\psi = \varphi - \frac{1}{2} \sum_{i=1}^h z_i \frac{\partial \varphi}{\partial z_i}$ and $\alpha_i = \frac{1}{2} \frac{\partial \varphi}{\partial z_i}$.

Proof. The relevant exterior differential system is given by

$$0 = \langle \omega, d\omega \rangle = \langle \omega, d\psi \cdot e_0 + \sum_{i=1}^h d\alpha_i \cdot e_i + \sum_{i=1}^h dz_i \cdot f_i \rangle = -d\psi - \sum_{i=1}^h z_i d\alpha_i + \sum_{i=1}^h \alpha_i dz_i.$$

It follows that $d\psi + \sum_{i=1}^h z_i d\alpha_i - \sum_{i=1}^h \alpha_i dz_i = 0$. Setting $t_0 = \psi + \sum_{i=1}^h z_i \alpha_i$ and $s_i = 2\alpha_i$ gives

$$\begin{aligned} dt_0 - \sum_{i=1}^h s_i dz_i &= d\psi + \sum_{i=1}^h z_i d\alpha_i + \sum_{i=1}^h \alpha_i dz_i - 2 \sum_{i=1}^h \alpha_i dz_i \\ &= d\psi + \sum_{i=1}^h z_i d\alpha_i - \sum_{i=1}^h \alpha_i dz_i. \end{aligned}$$

This is then solved by $t_0 = \varphi(z_1, \dots, z_h)$, an arbitrary function of the z_i , and $s_i = \frac{\partial \varphi}{\partial z_i}$. Explicitly, one gets the formula (4.5). \square

Additionally, a computation left to the reader shows that the cubic form is expressed as follows (see [Fri91, §5]).

Proposition 4.6. *With notations as above, the cubic form for Z_φ is*

$$(4.7) \quad \Xi \left(\frac{\partial}{\partial z_r}, \frac{\partial}{\partial z_s}, \frac{\partial}{\partial z_t} \right) = -\frac{1}{2} \frac{\partial^3 \varphi}{\partial z_r \partial z_s \partial z_t}.$$

In particular, Ξ is constant iff φ is a polynomial of degree at most 3. \square

Remark 4.8. Compare the above results with [Voi99, Lemma 3.4] which says $\frac{\partial \alpha_i}{\partial z_j} = \frac{\partial \alpha_j}{\partial z_i}$ in the notation of Equation (4.3). This guarantees the existence of a *potential* φ (denoted F in [Voi99, p. 42]). Then, Proposition 4.6 is just [Voi99, Prop. 3.3].

4.2. The global case. We now assume that (4.5) is actually a global (i.e. $z = (z_1, \dots, z_h) \in \mathbb{A}^h$ and φ is a polynomial in z) description of the horizontal subvariety $\hat{Z} \subseteq \check{\mathbf{D}}$. Suppose that the variety \hat{Z} is invariant under a maximally unipotent $T \in \mathrm{Sp}(2h+2, \mathbb{Z})$ preserving the filtration W_\bullet (and defining it in the sense of monodromy weight filtrations). It is easy to see that φ then satisfies a difference equation with terms which are polynomials in the z_i of degree at most two. For example, in case $h = 1$, with T given by $Tf_0 = f_0 + af_1 + be_1 + ce_0$, $Tf_1 = f_1 + de_1 + (b - ad)e_0$, $Te_1 = e_1 - ae_0$ and $Te_0 = e_0$, the maximal unipotency condition says that $ad \neq 0$, and the function φ satisfies the difference equation:

$$(4.9) \quad \varphi(z + a) - \varphi(z) = dz^2 + 2bz + (ab + c).$$

This is only possible in general if the polynomial φ has degree at most 3. A similar statement, under mild nondegeneracy assumptions or by considering instead a system of h commuting difference equations, is true in case $h > 1$ as well. Thus, it is natural to make the following assumption on φ :

Convention 4.10. The function φ of (4.5) is a **homogeneous** polynomial of degree 3.

In this case, by Euler's theorem, we can write

$$(4.11) \quad \omega = -\frac{1}{2}\varphi e_0 + \frac{1}{2} \sum_{i=1}^h \frac{\partial \varphi}{\partial z_i} e_i + \sum_{i=1}^h z_i f_i + f_0.$$

However, to avoid the factors of $\frac{1}{2}$, we replace $\frac{1}{2}\varphi$ by φ , and assume that \hat{Z} is described via

$$(4.12) \quad \omega = -\varphi e_0 + \sum_{i=1}^h \frac{\partial \varphi}{\partial z_i} e_i + \sum_{i=1}^h z_i f_i + f_0,$$

where again φ is a homogeneous polynomial of degree 3 in z_1, \dots, z_h . Conversely, if \hat{Z} is so defined, then a calculation shows that \hat{Z} is invariant under the (symplectic) transformation T_v defined by

$$(4.13) \quad \begin{aligned} T_v f_0 &= f_0 + \sum_i v_i f_i + \sum_i \frac{\partial \varphi}{\partial z_i}(v) e_i + (-\varphi(v)) e_0; \\ T_v f_i &= f_i + \sum_j \frac{\partial^2 \varphi}{\partial z_i \partial z_j}(v) e_j + \left(-\frac{\partial \varphi}{\partial z_i}(v) \right) e_0 \quad (i \neq 0); \\ T_v e_i &= e_i - v_i e_0 \quad (i \neq 0); \\ T_v e_0 &= e_0, \end{aligned}$$

because $T_v \omega(z) = \omega(z + v)$. Note that T_v is rational if φ has rational coefficients and $v \in \mathbb{Q}^h$, and similarly T_v is real if φ has real coefficients and $v \in \mathbb{R}^h$. Thus there is an action on \hat{Z} of an abelian unipotent group U isomorphic to \mathbb{G}_a^h . There is also a (non-symplectic) action of \mathbb{G}_m on \hat{Z} : given $\lambda \in \mathbb{G}_m$, the automorphism S_λ of \mathbb{P}^{2h+1} defined by $S_\lambda(e_0) = \lambda^3 e_0$, $S_\lambda(e_i) = \lambda^2 e_i$, $S_\lambda(f_i) = \lambda f_i$, and $S_\lambda(f_0) = f_0$ satisfies $S_\lambda \omega(z) = \omega(\lambda z)$. For φ general, these are in fact all of the automorphisms of \hat{Z} , as we shall see below.

Remark 4.14. (1) If instead of assuming that the subvariety \hat{Z} is invariant under a system of h commuting difference equations, we assume that \hat{Z} is invariant under an h -dimensional unipotent abelian subgroup of the complex symplectic group satisfying the appropriate conditions, then one can always choose a complex symplectic basis in which φ is homogeneous. This argument is essentially given in the course of the proof of Theorem 6.5 below.

- (2) With T_v defined as in Equation (4.13), consider $T_v - \text{Id}$, which differs from $N_v = \log T_v$ by an invertible matrix commuting with T_v and N_v , and let $C \subseteq \mathbb{P}^{h-1}$ be the cubic hypersurface $V(\varphi)$ defined by the homogeneous polynomial φ . Then $N_v^2 = 0, N_v^3 \neq 0 \iff v \neq 0$ and v defines a point \bar{v} of \mathbb{P}^{h-1} lying on $\text{Sing } C$, $N_v^2 = 0, N_v^3 = 0 \iff$ the point \bar{v} of \mathbb{P}^{h-1} lies on $C - \text{Sing } C$, and $N_v^3 \neq 0 \iff$ the point \bar{v} of \mathbb{P}^{h-1} does not lie on C .

Equation (4.12) can be interpreted as a map $F: \mathbb{A}^h \rightarrow \mathbb{A}^{2h+2}$ defined by $F(z) = \omega(z)$. Since the period is only defined up to scaling, we homogenize and obtain a rational map $\mathbb{P}^h \dashrightarrow \mathbb{P}^{2h+1}$ via:

$$F(z_1, \dots, z_h, t) = -\varphi(z) e_0 + t \sum_{i=1}^h \frac{\partial \varphi}{\partial z_i} e_i + t^2 \sum_{i=1}^h z_i f_i + t^3 f_0.$$

Thus F is a dominant rational map from \mathbb{P}^h to \hat{Z} , which fails to be defined at the codimension 2 subvariety $t = \varphi = 0$, the cubic hypersurface $C = V(\varphi)$ defined by the homogeneous cubic polynomial φ in the hyperplane $H \cong \mathbb{P}^{h-1} \subseteq \mathbb{P}^h$ defined by $t = 0$ and with homogeneous coordinates z_1, \dots, z_h . Note that F is regular on the affine open \mathbb{A}^h defined by $t \neq 0$, and that the projection of \mathbb{P}^{2h+1} onto \mathbb{P}^h defined by taking the coordinates corresponding to f_1, \dots, f_h, f_0 induces an isomorphism on the corresponding affine open subsets \mathbb{A}^h . In particular, if $h = 1$, F embeds \mathbb{P}^1 in \mathbb{P}^3 as a rational normal cubic.

Clearly the rational map F corresponds to a subseries of the complete linear series $|\mathcal{O}_{\mathbb{P}^h}(3) - C|$ of cubics on \mathbb{P}^h passing through C . Let X be the blowup of \mathbb{P}^h along C , with exceptional divisor E , ruled over C via $\rho: E \rightarrow C$. The proper transform H' of H is then exceptional, i.e. the normal bundle $\mathcal{O}_{H'}(H')$ corresponds to $\mathcal{O}_{\mathbb{P}^{h-1}}(-2)$. Let \bar{X} be the normal variety which is the contraction of H' and let $\bar{E} \subseteq \bar{X}$ be the image of E , i.e. the contraction of E along $H' \cap E$. Clearly $E \cong \mathbb{P}(\mathcal{O}_C(1) \oplus \mathcal{O}_C(3)) \cong \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(2))$ and $H' \cap E$ is the negative section.

Theorem 4.15. *With notations as above, suppose that C is smooth.*

- (i) *The complete linear system $|\mathcal{O}_{\mathbb{P}^h}(3) - C|$ defines a base point free linear system on X . The associated morphism φ blows down H' and embeds \bar{E} as the cone over the Veronese image of C under $\mathcal{O}_H(2)$.*
- (ii) *The subsystem of $|\mathcal{O}_{\mathbb{P}^h}(3) - C|$ induced by F , i.e. the linear system Σ of cubics on \mathbb{P}^h spanned by $\varphi(z)$, $t\partial\varphi/\partial z_i$, $t^2 z_i$, and t^3 , defines a base point free finite birational morphism $\bar{X} \rightarrow \hat{Z} \subseteq \mathbb{P}^{2h+1}$ which is an embedding on the affine open subset $\mathbb{A}^h = \mathbb{P}^h - H \cong \bar{X} - \bar{E}$.*

Proof. (i) is straightforward. For (ii), one can check by hand that Σ is base point free. It then automatically induces a finite morphism on \bar{X} since the induced linear system is a subsystem of a very ample linear system, and it is birational since it is an embedding on the open subset \mathbb{A}^h . \square

Theorem 4.16. *Let $h > 1$. If C is smooth, then the identity component of $\text{Aut } \hat{Z}$ is isomorphic to the identity component of the group of automorphisms of \mathbb{P}^h fixing the hyperplane H and acting as the identity on H , and hence is isomorphic to the semidirect product of $U \cong \mathbb{G}_a^h$ and \mathbb{G}_m .*

Proof. Since $\bar{X} \rightarrow \hat{Z}$ is a finite birational morphism, it identifies \bar{X} with the normalization of \hat{Z} . Hence every automorphism f of \hat{Z} lifts to an automorphism of \bar{X} which fixes the unique singular point and thus induces an automorphism of X , also denoted f , with $f^{-1}(H') = H'$. Since we are only considering the identity component, it follows that f^* is the identity on $\text{Pic } X$. Hence $f^{-1}(E)$ is an effective divisor linearly equivalent to E , and so $f^{-1}(E) = E$ since $f^{-1}(E)$ must contain every \mathbb{P}^1 fiber of the morphism $\rho: E \rightarrow C$. It follows that f induces and is induced by an automorphism of \mathbb{P}^h , which we continue to denote by f and now view as an element of $\text{PGL}(h+1)$, such that $f(H) = H$ and $f(C) = C$. As C is a cubic in $H \cong \mathbb{P}^{h-1}$ with $h > 1$, there are only finitely many automorphisms of H fixing C . Thus the restriction of f to H is the identity by connectedness. After normalizing f by a scalar, we can then assume that $f(z_1, \dots, z_h, t) = (z_1 + v_1 t, \dots, z_h + v_h t, \lambda t)$ as claimed. \square

Remark 4.17. The proof of Theorem 4.16 can be interpreted as saying that, in case C is smooth, the Legendrian submanifold $F(\mathbb{A}^h) \subseteq \mathbb{A}^{2h+1} \subseteq \mathbb{P}^{2h+1}$ defined

by $F(z) = \omega(z)$ cannot be completed to a (projective) Legendrian submanifold of \mathbb{P}^{2h+1} ; in fact, the normalization of the closure \hat{Z} has a unique singular point.

4.3. The Hodge–Riemann bilinear relations. We now discuss the inequalities imposed on C by the Riemann-Hodge bilinear relations. Some related results, from the point of view of the mirror manifold, have been given by Trenner–Wilson and Trenner [TW11], [Tre10].

We continue to use the normalization

$$\omega(z) = -\varphi(z)e_0 + \sum_{i=1}^h \frac{\partial \varphi}{\partial z_i} e_i + \sum_{i=1}^h z_i f_i + f_0,$$

and assume throughout this subsection that φ has **real** coefficients.

Theorem 4.18. *Let $z = (z_1, \dots, z_h) \in \mathbb{C}^h$ and let $y = (y_1, \dots, y_h) \in \mathbb{R}^h$, where $y_i = \text{Im } z_i$. Then the open set of $z \in \mathbb{C}^h$ where the Hodge–Riemann inequalities are satisfied is given by the set of $z \in \mathbb{C}^h$ such that $\varphi(y) < 0$ and the signature of the quadratic form $\left(\frac{\partial^2 \varphi}{\partial z_i \partial z_j} \right) (y)$ is $(h-1, 1)$.*

Proof. We shall just outline the argument and omit many of the calculations. The Hodge–Riemann inequalities are the statement that $\sqrt{-1} \langle \omega, \bar{\omega} \rangle > 0$ and that the form on $V^{2,1}$ given by $H(\psi) = \sqrt{-1} \langle \psi, \bar{\psi} \rangle$ is negative definite. Then the fact that z must satisfy $\varphi(y) < 0$ follows from:

$$(4.19) \quad \langle \omega, \bar{\omega} \rangle = 8\sqrt{-1}\varphi(y).$$

Next, to compute the sign on $V^{2,1}$, define

$$\omega_i = \frac{\partial \omega}{\partial z_i} = -\frac{\partial \varphi}{\partial z_i} e_0 + \sum_{j=1}^h \left(\frac{\partial^2 \varphi}{\partial z_i \partial z_j} \right) e_j + f_i.$$

Then a computation gives

$$(4.20) \quad \langle \omega_i, \bar{\omega}_j \rangle = 2\sqrt{-1} \frac{\partial^2 \varphi}{\partial z_i \partial z_j} (y).$$

We must modify ω_i by a multiple of ω to make it orthogonal to $\bar{\omega}$, and hence an element of $V^{2,1}$. To find the correct multiple, use:

$$(4.21) \quad \langle \omega_i, \bar{\omega} \rangle = -\frac{\partial \varphi}{\partial z_i} (z - \bar{z}) = -\frac{\partial \varphi}{\partial z_i} (2\sqrt{-1}y) = 4\frac{\partial \varphi}{\partial z_i} (y).$$

Using the skew symmetry and reality of the pairing, we have:

$$\langle \omega, \bar{\omega}_i \rangle = \overline{\langle \bar{\omega}, \omega_i \rangle} = -\overline{\langle \omega_i, \bar{\omega} \rangle} = -4\frac{\partial \varphi}{\partial z_i} (y).$$

Now let us choose a_i such that, with $\psi_i = a_i \omega + \omega_i$, we have $\langle \psi_i, \bar{\omega} \rangle = 0$, i.e. $\psi_i \in V^{2,1}$. Writing this as $0 = \langle \psi_i, \bar{\omega} \rangle = a_i \langle \omega, \bar{\omega} \rangle + \langle \omega_i, \bar{\omega} \rangle$, we see that

$$a_i = -\frac{\langle \omega_i, \bar{\omega} \rangle}{\langle \omega, \bar{\omega} \rangle}.$$

We now work out the signature of the intersection matrix $(\langle \psi_i, \bar{\psi}_j \rangle)$. We shall use the shorthand $\varphi_i = \frac{\partial \varphi}{\partial z_i}$ and $\varphi_{ij} = \frac{\partial^2 \varphi}{\partial z_i \partial z_j}$. Then a calculation gives

$$(4.22) \quad \langle \psi_i, \bar{\psi}_j \rangle = \frac{2\sqrt{-1}}{\varphi(y)} (-\varphi_i(y)\varphi_j(y) + \varphi(y)\varphi_{ij}(y)).$$

We could also write this as

$$\langle \psi_i, \bar{\psi}_j \rangle = \frac{2\sqrt{-1}}{\varphi(y)} \frac{\partial^2}{\partial z_i \partial z_j} \log \varphi(y).$$

Now, we want the Hermitian matrix $\sqrt{-1}(\langle \psi_i, \bar{\psi}_j \rangle) = -\frac{2}{\varphi(y)}(-\varphi_i(y)\varphi_j(y) + \varphi(y)\varphi_{ij}(y))$ to be negative definite, and hence, since we have already assume $\varphi(y) < 0$, we want the real symmetric matrix $(-\varphi_i(y)\varphi_j(y) + \varphi(y)\varphi_{ij}(y))$ to be negative definite.

Let $B(x, y) = \sum_{i=1}^h x_i y_i$ be the standard inner product on \mathbb{R}^h and let Υ and Υ_j be the vectors defined by

$$\Upsilon(y) = (\varphi_1(y), \dots, \varphi_h(y)); \quad \Upsilon_j(y) = (\varphi_{1j}(y), \dots, \varphi_{hj}(y)).$$

Then, by Euler's theorem, we have

$$B(y, \Upsilon(y)) = 3\varphi(y); \quad B(y, \Upsilon_j(y)) = 2\varphi_j(y).$$

In particular, we see that y is not orthogonal to $\Upsilon(y)$ with respect to B , and hence that every vector in \mathbb{R}^n can be uniquely written as $ty + v$ for some $t \in \mathbb{R}$, where $v \in \Upsilon^\perp$ (the perpendicular space for the standard inner product). Also, for a vector $(v_1, \dots, v_h) \in \mathbb{R}^h$,

$${}^t v(\varphi_i(y)\varphi_j(y))v = (B(v, \Upsilon(y)))^2.$$

Hence, if $v \in \Upsilon^\perp$, then

$${}^t v(-\varphi_i(y)\varphi_j(y) + \varphi(y)\varphi_{ij}(y))v = {}^t v(\varphi(y)\varphi_{ij}(y))v.$$

Now ${}^t y(\varphi(y)\varphi_{ij}(y))y = 6\varphi^2(y) > 0$, and

$$\begin{aligned} {}^t y(-\varphi_i(y)\varphi_j(y) + \varphi(y)\varphi_{ij}(y))y &= -(B(y, \Upsilon(y)))^2 + \varphi(y)B(y, 2\Upsilon(y)) \\ &= -9\varphi^2(y) + 6\varphi^2(y) = -3\varphi^2(y) < 0. \end{aligned}$$

Note that, if $v \in \Upsilon^\perp$, ${}^t v(\varphi_{ij})y = 2B(v, \Upsilon(y)) = 0$, and likewise ${}^t v(\varphi_i \varphi_j)y = 3\varphi(y)B(v, \Upsilon(y)) = 0$. Hence Υ^\perp is contained in the orthogonal space to y corresponding to the quadratic form (φ_{ij}) (or to $(\varphi(y)\varphi_{ij})$) and to $(-\varphi_i(y)\varphi_j(y))$. Thus, applying the form $(-\varphi_i(y)\varphi_j(y) + \varphi(y)\varphi_{ij}(y))$ to a vector of the form $ty + v$ with $t \in \mathbb{R}$ and $v \in \Upsilon^\perp$ gives

$${}^t (ty + v)(-\varphi_i(y)\varphi_j(y) + \varphi(y)\varphi_{ij}(y))(ty + v) = -3t^2\varphi^2(y) + {}^t v(\varphi(y)\varphi_{ij}(y))v.$$

Now, if the signature of the matrix $(\varphi(y)\varphi_{ij}(y))$ is $(1, h-1)$, then $(\varphi(y)\varphi_{ij}(y))$ is negative definite on Υ^\perp and hence

$${}^t (ty + v)(-\varphi_i(y)\varphi_j(y) + \varphi(y)\varphi_{ij}(y))(ty + v) \leq 0,$$

with equality $\iff t = v = 0$. Conversely, if $(-\varphi_i(y)\varphi_j(y) + \varphi(y)\varphi_{ij}(y))$ is negative definite, then since the induced quadratic form agrees with that for $(\varphi(y)\varphi_{ij}(y))$ on Υ^\perp , $(\varphi(y)\varphi_{ij}(y))$ has at least $h-1$ negative eigenvalues, and since it has at least one positive eigenvalue corresponding to y , its signature is $(1, h-1)$. \square

Remark 4.23. Let \hat{Z} be the subvariety of \mathbb{P}^{2h+1} defined by Equation (4.12) and let Z be the open subset of \hat{Z} defined by the Hodge–Riemann inequalities. Clearly, in case $\varphi = 0$, $Z = \emptyset$. In all other cases it is nonempty: in fact, if L is a line through the origin in \mathbb{C}^h defined over \mathbb{R} and such that $\varphi|_L \neq 0$, and $f: L \rightarrow \hat{Z}$ is the natural morphism, then $f^{-1}(Z)$ is an upper half plane \mathfrak{H} embedded in its compact dual, which is isomorphic to \mathbb{P}^1 .

5. THE COMPLEX CASE

The description of Equation (4.12) does not suffice to handle the case where $\varphi = 0$ (the unit ball case, see Remark 2.35). Also, in certain situations we would like to relax the non-degeneracy condition that the derivatives $\partial\omega/\partial z_i$ span all of F^2/F^3 . To describe the relevant horizontal subvarieties which we shall encounter, we fix the following notation:

Notation 5.1. Let \mathbb{C}^{2h+2} have a complex basis $\varepsilon_0, \dots, \varepsilon_h, \delta_0, \dots, \delta_h$ and a real structure such that $\bar{\varepsilon}_i = \delta_i$. Let $\langle \cdot, \cdot \rangle$ be the unique symplectic form on \mathbb{C}^{2h+2} such that $\langle \varepsilon_i, \varepsilon_j \rangle = \langle \delta_i, \delta_j \rangle = 0$ for all i and j and $\langle \varepsilon_0, \delta_0 \rangle = 2\sqrt{-1}$, $\langle \varepsilon_i, \delta_i \rangle = -2\sqrt{-1}$ for $1 \leq i \leq a$, and $\langle \varepsilon_i, \delta_i \rangle = 2\sqrt{-1}$ for $a+1 \leq i \leq h$ (where we allow the possibility that $a = h$ and $b = 0$). Note that the ε_i span a maximal isotropic complex subspace of \mathbb{C}^{2h+2} , as do the δ_i .

More invariantly, we suppose that V_+ is an $(h+1)$ -dimensional complex vector space with a non-degenerate Hermitian inner product $[\cdot, \cdot]$, which thus gives a complex anti-linear isomorphism f from V_+ to V_- , where V_- the dual of V_+ . The isomorphism f then defines a real structure on $V_{\mathbb{C}} := V_+ \oplus V_-$ (with $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$); for example, if $v \in V_+$, $\operatorname{Re} v = \frac{1}{2}(v + f(v))$ and $\operatorname{Im} v = \frac{1}{2\sqrt{-1}}(v - f(v)) = \operatorname{Re}(-\sqrt{-1}v)$. If $\varepsilon_0, \dots, \varepsilon_h$ is a diagonal basis for V_+ with respect to the form $[\cdot, \cdot]$, i.e. $[\varepsilon_i, \varepsilon_j] = (-1)^{a_i} \delta_{ij}$ with $a_i = 0$ or 1 , and we set $\delta_i = \bar{\varepsilon}_i$, then $\delta_i = (-1)^{a_i} \varepsilon_i^*$, where $\varepsilon_0^*, \dots, \varepsilon_h^*$ is the dual basis for $V_- \cong V_+^{\vee}$. On $V_{\mathbb{C}}$, there is the natural symplectic form

$$\langle v_1 + \xi_1, v_2 + \xi_2 \rangle_0 = \xi_2(v_1) - \xi_1(v_2).$$

Applying the form to a pair of real vectors gives

$$\left\langle \frac{1}{2}(v + f(v)), \frac{1}{2}(w + f(w)) \right\rangle_0 = \frac{1}{4}(f(w)(v) - f(v)(w)) = -\frac{\sqrt{-1}}{2} \operatorname{Im}[w, v].$$

Thus $\langle \cdot, \cdot \rangle = 2\sqrt{-1} \langle \cdot, \cdot \rangle_0$ is a real symplectic form on V , with the property that

$$\langle \varepsilon_i, \delta_i \rangle = \langle \varepsilon_i, (-1)^{a_i} \varepsilon_i^* \rangle = (-1)^{a_i} 2\sqrt{-1}.$$

Conversely, with notation as at the beginning of this section, we define $V_+ = \operatorname{span}\{\delta_0, \dots, \delta_h\}$ and $V_- = \operatorname{span}\{\varepsilon_0, \dots, \varepsilon_h\}$ and use the symplectic pairing to define an identification of V_- with the dual of V_+ , and define a complex anti-linear isomorphism from V_+ to V_- by taking the Hermitian form

$$[v, w] = \frac{1}{2\sqrt{-1}} \langle \bar{v}, w \rangle.$$

5.1. The non-degenerate case. In this case, we let $a = h$ and $b = 0$ and work out the analogue of the preceding section using the complex basis $\varepsilon_0, \dots, \varepsilon_h, \delta_0, \dots, \delta_h$. Writing

$$\omega(z) = \Psi(z)\varepsilon_0 + \sum_{i=1}^h A_i(z)\varepsilon_i + \sum_{i=1}^h z_i\delta_i + \delta_0,$$

the only difference between this picture and that of Equation (4.5) is the sign change between $\langle \varepsilon_0, \delta_0 \rangle$ and $\langle \varepsilon_i, \delta_i \rangle$ for $i > 0$, giving: there exists a function Φ such that

$$(5.2) \quad \omega(z) = \left(\Phi - \frac{1}{2} \sum_{i=1}^h z_i \frac{\partial \Phi}{\partial z_i} \right) \varepsilon_0 - \frac{1}{2} \sum_{i=1}^h \frac{\partial \Phi}{\partial z_i} \varepsilon_i + \sum_{i=1}^h z_i \delta_i + \delta_0.$$

This determines a filtration F^\bullet of \mathbb{C}^{2h+2} , by taking

$$F^3(z) = \mathbb{C} \cdot \omega(z) \subseteq F^2(z) = \text{span} \left\{ \omega(z), \frac{\partial \omega}{\partial z_1}, \dots, \frac{\partial \omega}{\partial z_h} \right\},$$

and then setting $F^1(z) = (F^3(z))^\perp$ and $F^0 = V_{\mathbb{C}} \cong \mathbb{C}^{2h+2}$.

Remark 5.3. Note that, in case Φ is a homogeneous cubic polynomial, then $\Psi = -\frac{1}{2}\Phi$ as before. Normalizing again to eliminate the factor of 1/2 then gives

$$(5.4) \quad \omega(z) = -\Phi\varepsilon_0 - \sum_{i=1}^h \frac{\partial \Phi}{\partial z_i} \varepsilon_i + \sum_{i=1}^h z_i \delta_i + \delta_0.$$

If Φ in Equation 5.4 is 0, we are in the case of the unit ball, as we shall see below. If Φ is a homogeneous linear or quadratic polynomial, the complex variation of Hodge structure defined by Equation 5.4 reduces to the case of the unit ball via an appropriate symplectic transformation.

For a general choice of the cubic Φ , it follows from Theorem 4.16 that a horizontal subvariety \hat{Z} cannot be simultaneously expressed in the form Equation (5.4) and the form Equation (4.5) for two different choices of bases $\varepsilon_0, \dots, \varepsilon_h, \delta_0, \dots, \delta_h$ and $e_0, \dots, e_h, f_0, \dots, f_h$. However, as we shall see in the next section, this is always possible in the Hermitian symmetric tube domain case.

If Φ and $\partial\Phi/\partial z_i$ both vanish at the origin, then the Hodge structure at 0 is determined by $F^3 = \mathbb{C} \cdot \delta_0 \subseteq F^2 = \text{span}\{\delta_0, \delta_1, \dots, \delta_h\} = V_+$, and hence $V^{3,0}(0) = \mathbb{C} \cdot \delta_0$, $V^{2,1}(0) = \text{span}\{\delta_1, \dots, \delta_h\}$, $V^{1,2}(0) = \text{span}\{\varepsilon_1, \dots, \varepsilon_h\}$, and $V^{0,3}(0) = \mathbb{C} \cdot \varepsilon_0$. Thus the filtration automatically satisfies the Hodge–Riemann inequalities, which was the reason for our choice of signs. In particular, the open subset Z of \hat{Z} is always nonempty in this case. We will not write out the Hodge–Riemann inequalities in general, although this is straightforward to do, except to note that it is easy to check directly that, in case Φ is identically 0, the inequalities amount to:

$$\sum_{i=1}^h |z_i|^2 < 1,$$

i.e. the horizontal subvariety in question is the h -dimensional complex unit ball.

5.2. A degenerate case. Here we are interested in the case $a < h$ and hence $b > 0$. We define a holomorphically varying line in \mathbb{P}^{2h+1} , the analogue in some sense of Equation (4.5), via

$$(5.5) \quad \omega(z) = \sum_{k=a+1}^h q_k(z) \delta_k + \sum_{i=1}^a z_i \delta_i + \delta_0.$$

Taking derivatives, we have, for $1 \leq i \leq a$,

$$\omega_i(z) = \frac{\partial \omega}{\partial z_i} = \sum_{k=a+1}^h \frac{\partial q_k}{\partial z_i} \delta_k + \delta_i.$$

To complete this to a Hodge filtration, we define

$$F^2 = \text{span}\{\omega(z), \omega_1(z), \dots, \omega_a(z)\} \oplus \text{span}\{\omega(z), \omega_1(z), \dots, \omega_a(z)\}^\perp,$$

where $\text{span}\{\omega(z), \omega_1(z), \dots, \omega_a(z)\}^\perp$ denotes the orthogonal space in W^\vee . Explicitly, for $a+1 \leq k \leq h$, let

$$\omega_k(z) = \varepsilon_k + \sum_{i=1}^a \frac{\partial q_k}{\partial z_i} \varepsilon_i + s_k \varepsilon_0,$$

where we set

$$s_k = \sum_{i=1}^a z_i \frac{\partial q_k}{\partial z_i} - q_k.$$

Note in particular that if q_k is a homogeneous quadratic polynomial for all k , then $s_k = q_k$.

Then $\omega_{a+1}(z), \dots, \omega_h(z)$ is a basis for $\text{span}\{\omega(z), \omega_1(z), \dots, \omega_a(z)\}^\perp$. It follows that $\omega(z), \omega_1(z), \dots, \omega_h(z)$ span an isotropic subspace F^2 of $V_{\mathbb{C}} \cong \mathbb{C}^{2h+2}$. Setting $F^1 = (\omega)^\perp$ (under the symplectic form $\langle \cdot, \cdot \rangle$) then defines a weight three complex variation of Hodge structure $F^3 \subseteq F^2 \subseteq F^1 \subseteq F^0 = V_{\mathbb{C}}$ of CY type.

Remark 5.6. In case $b > 0$, the above variation of Hodge structure is never maximal, and in fact can be embedded in an h -dimensional complex variation of Hodge structure given by a slight variant of Equation (5.2). The simplest such choice is as follows: for $z = (z_1, \dots, z_h)$, setting

$$\begin{aligned} \Phi_0(z) &= -2 \sum_{k=a+1}^h z_k q_k(z_1, \dots, z_a); \\ \Psi_0(z) &= \Phi_0(z) - \frac{1}{2} \sum_{i=1}^h z_i \frac{\partial \Phi_0}{\partial z_i}; \\ A_i(z) &= \frac{1}{2} \frac{\partial \Phi_0}{\partial z_i} = \begin{cases} -\sum_{k=a+1}^h z_k \frac{\partial q_k}{\partial z_i}, & \text{if } i \leq a; \\ -q_i, & \text{if } i \geq a+1, \end{cases} \end{aligned}$$

and defining

$$(5.7) \quad \omega(z) = \Psi_0(z) \varepsilon_0 - \sum_{i=1}^a A_i(z) \varepsilon_i - \sum_{k=a+1}^h A_k(z) \delta_k + \sum_{i=1}^a z_i \delta_i + \sum_{k=a+1}^h z_k \varepsilon_k + \delta_0,$$

then we get a variation of Hodge structure of the form of Equation (5.2) which specializes to Equation (5.5) when $z_{a+1} = \dots = z_h = 0$. Here, we are free to replace Φ_0 by $\Phi = \Phi_0 + \Phi_1$, where Φ_1 is any polynomial in z_1, \dots, z_h which vanishes to order

at least two in z_{a+1}, \dots, z_h . Note that, in case the q_k are all homogeneous quadratic polynomials, then Φ_0 is a homogeneous cubic polynomial and $\Psi_0 = -\frac{1}{2}\Phi_0$. In this case, if we require that Φ_1 is also a homogeneous cubic polynomial vanishing to order at least two in z_{a+1}, \dots, z_h , then $\Phi = \Phi_0 + \Phi_1$ is a homogeneous cubic containing the linear space $z_{a+1} = \dots = z_h = 0$.

If $q_k(0) = \frac{\partial q_k}{\partial z_i} = 0$ for all i and k , then the Hodge filtration at 0 is determined by $F^3 = V^{3,0}(0) = \mathbb{C} \cdot \delta_0 \subseteq F^2 = \text{span}\{\delta_0, \delta_1, \dots, \delta_a, \varepsilon_{a+1}, \dots, \varepsilon_h\}$. Again, by our choice of signs, this satisfies the Hodge–Riemann inequalities. In practice, the q_j will be homogeneous quadratic functions of z . Note that the Hodge–Riemann inequalities for the degenerate case given by Equation (5.4) follow from those for the larger family given by Equation (5.7).

6. THE WEIGHT THREE HERMITIAN SYMMETRIC CASE

Our goal now is to show that the Hermitian symmetric weight three Calabi–Yau examples can be described as in the previous two sections, and to discuss rationality issues. We begin by recalling some standard facts about Hermitian symmetric spaces (see also §2.1). If G is a simple real algebraic group with maximal compact subgroup K such that $\mathcal{D} = G(\mathbb{R})/K$ is a Hermitian symmetric space, then the real Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ decomposes into the $+1$ and -1 eigenspaces of the Cartan involution. Moreover, the complexification $\mathfrak{p}_{\mathbb{C}}$ of \mathfrak{p} has a direct sum decomposition $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$, where each subspace \mathfrak{p}_{\pm} is an abelian Lie algebra corresponding to a unipotent subgroup P_{\pm} of $H(\mathbb{C})$. Then $\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_+$ is the Lie algebra of a parabolic subgroup $P_0(\mathbb{C}) = K(\mathbb{C})P_+$ defined over \mathbb{C} . Note that $G(\mathbb{C})/P_0(\mathbb{C})$ is the compact dual of \mathcal{D} , in the sense that $P_0(\mathbb{C}) \cap G(\mathbb{R}) = K$ and thus there is an inclusion $\mathcal{D} = G(\mathbb{R})/K \subseteq \tilde{\mathcal{D}} = G(\mathbb{C})/P_0(\mathbb{C})$. By the Borel and Harish-Chandra embedding theorem, $G(\mathbb{R}) \subseteq P_-K(\mathbb{C})P_+ = P_-P_0(\mathbb{C})$ and \mathcal{D} is embedded in the open subset $P_-P_0(\mathbb{C})$ of $\tilde{\mathcal{D}}$. Clearly, P_+ is the unipotent radical of $P_0(\mathbb{C})$, $K(\mathbb{C})$ is the Levi subgroup, and $K(\mathbb{C})P_-$ is the opposite parabolic, with unipotent radical P_- .

In terms of Hodge structures associated to a representation of a reductive form of G , having chosen a reference Hodge structure F_0^\bullet , K is the stabilizer of F_0^\bullet in $G(\mathbb{R})$ and $P_0(\mathbb{C})$ is the stabilizer of F_0^\bullet in $G(\mathbb{C})$. Hence every Hodge structure is in the P_- -orbit of F_0^\bullet .

Lemma 6.1. *The parabolic subgroup $P_0(\mathbb{C})$ is conjugate in $G(\mathbb{C})$ to a parabolic subgroup $P(\mathbb{C})$, where P is any maximal real parabolic subgroup corresponding to a zero-dimensional boundary component.*

Proof. By a theorem of Korányi–Wolf [KW65, Theorem 5.9] (and the remark at the end of Section §5 in op. cit.), the subgroup $P = (\text{Ad}(c))^{-1}P_0(\mathbb{C}) \cap G(\mathbb{R})$ is a real parabolic subgroup of G , where $c = c_r$ is the Cayley transform (which is all that we shall need in this section). The more precise identification of P as the maximal parabolic subgroup corresponding to a zero-dimensional boundary component is given in [WK65, Corollary 6.9]. \square

Let P be a maximal real parabolic subgroup corresponding to a 0-dimensional boundary component of \mathcal{D} . Then its unipotent radical U satisfies: the complexification $U(\mathbb{C})$ is conjugate in $G(\mathbb{C})$ to P_- . Note that, if k is a subfield of \mathbb{R} such that H and P are defined over k , then U is also defined over k .

Remark 6.2. Suppose that $T \in G(\mathbb{Z})$ is a monodromy matrix corresponding to a holomorphic map $\Delta^* \rightarrow \mathcal{D}/G(\mathbb{Z})$. Then T is conjugate in $G(\mathbb{R})$ to an element of $U(F)$, where $U(F)$ is the center of the unipotent radical of the real parabolic subgroup corresponding to an appropriate rational boundary component F (see for example [AMRT75, Theorem, p. 279] or [Sch74, Satz 12]). One can show (cf. [AMRT75, Theorem 3, p. 240]) that, for such a T , the element T is conjugate in $G(\mathbb{R})$ to an element of U , the unipotent radical of a maximal real parabolic subgroup corresponding to a 0-dimensional boundary component, and hence in $G(\mathbb{C})$ to an element of P_- .

Finally, we shall use the following:

Lemma 6.3. *Let $\iota: \check{\mathbf{D}} \rightarrow \mathbb{P}^{2h+1}$ be the morphism defined by taking the complex line F^3 and let \mathcal{D} be a Hermitian subvariety of \mathbf{D} . Then ι induces an embedding $\mathcal{D} \rightarrow \mathbb{P}^{2h+1}$.*

Proof. Given $o_1, o_2 \in \mathcal{D}$, with x_1, x_2 the corresponding points of \mathbb{P}^{2h+1} , suppose that $x_1 = x_2$. View the maximal compact subgroup K as the stabilizer of o_1 . Then P_0 is the stabilizer of the point $x_1 \in \mathbb{P}^{2h+1}$ (as one can easily check; see also Remark 2.15). There exists a $g \in G(\mathbb{R})$ such that $go_1 = o_2$, and hence $g \cdot x_1 = x_2 = x_1$. It follows that $g \in G(\mathbb{R}) \cap P_0 = K$, and hence $o_1 = o_2$. \square

6.1. Tube domain case. In the tube domain case, we can give a very explicit description of the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and of the minuscule representation $V_{\mathbb{C}}$ of $G(\mathbb{C})$ corresponding to the weight three variation of Hodge structure (see Lemma 2.27 and Corollary 2.34). This discussion is closely related to some of the work of Landsberg and Manivel (esp. [LM01] and [LM07]).

In the tube domain case, $\mathfrak{k}_{\mathbb{C}} \cong \mathfrak{g}'_{\mathbb{C}} \oplus \mathfrak{z}$, where $\mathfrak{g}'_{\mathbb{C}}$ is a simple (complex) Lie algebra and, as a $\mathfrak{k}_{\mathbb{C}}$ -module, $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{k}_{\mathbb{C}} \oplus W \oplus W^{\vee}$, where W is an irreducible representation of $\mathfrak{k}_{\mathbb{C}}$ which is a minuscule representation of $\mathfrak{g}'_{\mathbb{C}}$, and W^{\vee} is its dual, and in fact $W = \mathfrak{p}_-$, $W^{\vee} = \mathfrak{p}_+$ in the notation above. There is also a $\mathfrak{g}'_{\mathbb{C}}$ -invariant linear map $B: \text{Sym}^2 W \rightarrow W^{\vee}$ such that the associated cubic tensor $C(w_1, w_2, w_3) = \langle B(w_1, w_2), w_3 \rangle$ is nonzero, $\mathfrak{g}'_{\mathbb{C}}$ -invariant, and symmetric, where $\langle \cdot, \cdot \rangle$ is the evaluation pairing $W \otimes W^{\vee} \rightarrow \mathbb{C}$. The minuscule representation $V_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$ splits as a $\mathfrak{k}_{\mathbb{C}}$ -module as

$$(6.4) \quad V_{\mathbb{C}} \cong \underline{\mathbb{C}} \oplus W \oplus W^{\vee} \oplus \underline{\mathbb{C}},$$

where $\mathfrak{g}'_{\mathbb{C}}$ acts in the standard way on W and W^{\vee} and trivially on the two factors $\underline{\mathbb{C}}$, and the center $\mathfrak{z} = \mathbb{C} \cdot H_0$ (in the notation of §2.1) acts diagonally with weights $-3/2, -1/2, 1/2, 3/2$ respectively (compare to §2.2). In particular, V has a natural real structure, and complex conjugation exchanges W and W^{\vee} and the two factors $\underline{\mathbb{C}}$. In terms of Hodge structures, with K the stabilizer of a reference Hodge structure, this says that the first factor $\underline{\mathbb{C}} = V^{3,0}$, $W = V^{2,1}$, $W^{\vee} = V^{1,2}$, and the last factor $\underline{\mathbb{C}} = V^{0,3}$. An interesting example here is the E_7 case: V is the unique minuscule E_7 representation, the semi-simple part of K is E_6 , and W and W^{\vee} are the two minuscule representations of E_6 .

We are interested in the action of the space W on V . We will normalize the symplectic form on V (which is unique up to a scalar) to be given by

$$\langle (t_1, v_1, \xi_1, s_1), (t_2, v_2, \xi_2, s_2) \rangle = s_1 t_2 - s_2 t_1 + \langle \xi_1, v_2 \rangle - \langle \xi_2, v_1 \rangle.$$

The W -action is given by $w \in W \mapsto N_w$, where

$$N_w(t, v, \xi, s) = (0, tw, B(w, v), -\langle \xi, w \rangle).$$

Here the fact that $[N_{w_1}, N_{w_2}] = 0$ and that N_w preserves the symplectic form, i.e. that $\langle N_w V_1, V_2 \rangle = -\langle V_1, N_w V_2 \rangle = \langle N_w V_2, V_1 \rangle$ follow from the symmetry of $C(w_1, w_2, w_3) = \langle B(w_1, w_2), w_3 \rangle$. Note that

$$\begin{aligned} N_w^2(t, v, \xi, s) &= (0, 0, tB(w, w), -\langle B(w, v), w \rangle), \\ N_w^3(t, v, \xi, s) &= (0, 0, 0, -t\langle B(w, w), w \rangle), \end{aligned}$$

and hence $(\exp N_w)(t, v, \xi, s)$ is given by

$$\left(t, v + tw, \xi + B(w, v) + \frac{t}{2}B(w, w), s - \langle \xi, w \rangle - \frac{1}{2}\langle B(w, v), w \rangle - \frac{t}{6}\langle B(w, w), w \rangle \right).$$

In particular,

$$(\exp N_w)(1, 0, 0, 0) = \left(1, w, \frac{1}{2}B(w, w), -\frac{1}{6}C(w, w, w) \right).$$

This is in the form of Equation (5.4) with $\Phi(z) = -\frac{1}{6}C(w, w, w)$, for an appropriate complex basis $\{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_h, \delta_1, \dots, \delta_h, \delta_0\}$ and the cubic form φ is only defined over \mathbb{C} . (Note that in Equation (5.4) there is a minus sign in front of the factor $\frac{1}{2}B(w, w)$ since, up to the common factor of $\sqrt{-1}$, $\varepsilon_1, \dots, \varepsilon_h$ are the negatives of the dual basis for $\delta_1, \dots, \delta_h$.) We are more interested in the variant corresponding to Equation (4.12) where everything is defined over \mathbb{R} or over \mathbb{Q} , as follows:

Theorem 6.5. *Let k be a subfield of \mathbb{R} . Suppose that G is defined over k and that the maximal real parabolic subgroup P is also defined over k , so that its unipotent radical U is defined over k as well. Then there exists a symplectic k -basis $e_0, e_1, \dots, e_h, f_1, \dots, f_h, f_0$ of V and a cubic polynomial $\varphi(z_1, \dots, z_h)$ defined over k , such that \mathcal{D} is biholomorphic to the open subset of \mathbb{P}^{2h+2} given by lines*

$$\omega = -\varphi e_0 + \sum_{i=1}^h \frac{\partial \varphi}{\partial z_i} e_i + \sum_{i=1}^h z_i f_i + f_0$$

satisfying the Hodge–Riemann inequalities (see Theorem 4.18), and the action of U is given by the unipotent subgroup corresponding to $z \mapsto z + v$.

Proof. The abelian unipotent subgroup U defines a filtration V^\bullet of V , defined over k : if \mathfrak{u} is the Lie algebra of U , then $V^0 = V$, $V^1 = \mathfrak{u}(V)$, $V^2 = \mathfrak{u}(V^1)$, and $V^3 = \mathfrak{u}(V^2)$. Of course, the filtration V^\bullet is isomorphic over \mathbb{C} to the filtration defined by the action of W on V described above. In particular, $\dim V^0/V^1 = \dim V^3 = 1$ and $\dim V^1/V^2 = \dim V^2/V^3 = h$, and the symplectic form pairs V^0/V^1 with V^3 and V^1/V^2 with V^2/V^3 . The group P preserves the filtration V^\bullet , and in fact P is the stabilizer in G of V^3 . For $w \in \mathfrak{u}$, if N_w is the corresponding nilpotent endomorphism of V , if f_0 is a nonzero element of V^0/V^1 and $e_0 \in V^3$ is the dual element, then $w \mapsto N_w(f_0)$ defines a k -linear isomorphism $\mathfrak{u} \rightarrow V^1/V^2$ and hence identifies V^2/V^3 with \mathfrak{u}^\vee over k . Write $N_{w_1}N_{w_2}N_{w_3}(f_0) = C_0(w_1, w_2, w_3)e_0$, where C_0 is a symmetric trilinear form defined over k . Similarly, the linear map $w \mapsto N_w N_v(e_0) \bmod V^3$ defines a symmetric bilinear form $B_0: \mathfrak{u} \otimes \mathfrak{u} \rightarrow \mathfrak{u}^\vee$, and $C_0(w_1, w_2, w_3) = \langle B_0(w_1, w_2), w_3 \rangle$ from the definition. Note that, over \mathbb{C} , C_0 and B_0 are projectively equivalent to C and B as defined above.

Since $U(\mathbb{C})$ is conjugate to P_- , there exists a point $o \in G(\mathbb{C})/P(\mathbb{C})$ such that $U(\mathbb{C}) \cdot o$ is a nonempty Zariski open subset of $G(\mathbb{C})/P(\mathbb{C})$. Since $G(k)$ is Zariski dense in $G(\mathbb{C})$, $U(\mathbb{C}) \cdot o$ contains a k -rational point. Hence we can assume that o itself is k -rational and thus that $U(\mathbb{C}) \cdot o$ is an affine space defined over k . Since $G(\mathbb{C})/P(\mathbb{C})$ is a Legendrian subvariety of the space $\check{\mathbf{D}}$ of isotropic filtrations of V in the usual sense, by taking the first subspace $\mathbb{C} \cdot \omega$ of the filtration we have a morphism from $U(\mathbb{C}) \cdot o$ to the k -rational variety $U(\mathbb{C}) \cdot x$, where $x \in \mathbb{P}^{2h+2}(k)$ is the line spanned by ω . Write $x = \mathbb{C} \cdot f_0$ where $f_0 \in V(k)$. We now define subspaces of V , defined over k , as follows:

$$\begin{aligned} V_0 &= \mathbb{C} \cdot f_0; \\ V_1 &= \mathfrak{u}(V_0); \\ V_2 &= V^2 \cap (f_0)^\perp; \\ V_3 &= V^3 = \mathbb{C} \cdot e_0, \end{aligned}$$

where $\langle e_0, f_0 \rangle = 1$ and e_0 is uniquely defined up to scalar. By construction, \mathfrak{u} is the tangent space to $U(\mathbb{C}) \cdot o$ at every point, so by horizontality $V_0 \oplus V_1$ is an isotropic subspace of V . By definition, $\mathfrak{u}(V_0) = V_1$ and in fact the choice of f_0 identifies V_1 with \mathfrak{u} . Moreover,

$$\langle \mathfrak{u}(V_1), f_0 \rangle = \langle \mathfrak{u}(f_0), V_1 \rangle = \{0\}$$

since V_1 is isotropic. Hence $\mathfrak{u}(V_1) \subseteq V^2 \cap (f_0)^\perp = V_2$. Finally, $\mathfrak{u}(V_2) \subseteq V^3 = V_3$. Clearly, $V^2 = V_2 \oplus V_3$ and V^2 is isotropic since $V^2 = \text{Im } \mathfrak{u}^2$. It is then easy to see that $V = V_0 \oplus V_1 \oplus V_2 \oplus V_3$.

Writing N_w in terms of the direct sum decomposition and the identifications above gives

$$N_w(t, v, \xi, s) = (0, tw, B_0(v, w), -\langle \xi, w \rangle),$$

where the last entry follows from the fact that N_w preserves the symplectic form and the identification of V_1 with \mathfrak{u} and of V_2 with \mathfrak{u}^\vee , via

$$\langle N_w(\xi), f_0 \rangle = \langle N_w(f_0), \xi \rangle = \langle w, \xi \rangle = -\langle \xi, w \rangle.$$

Exponentiating the action of N_w shows that the affine space $\mathbb{A} = U(\mathbb{C}) \cdot x$ is described via Equation 4.12. Moreover, the Zariski closure $\overline{\mathbb{A}}$ is equal to the image of $\check{\mathcal{D}}$ and hence certainly contains the image of \mathcal{D} . We must show that \mathcal{D} is contained in \mathbb{A} , not just in its closure. Note that, using the notation of Equation 4.12 and taking homogeneous coordinates $x_0, x_1, \dots, x_h, y_1, \dots, y_h, y_0$ on \mathbb{P}^{2h+1} corresponding to the basis $e_0, e_1, \dots, e_h, f_1, \dots, f_h, f_0$, the projective closure $\overline{\mathbb{A}}$ is contained in the variety defined by the homogeneous equations $y_0^2 x_0 = -\varphi(y_1, \dots, y_h)$ and $y_0 x_i = \frac{\partial \varphi}{\partial z_i}(y_1, \dots, y_h)$ for $i = 1, \dots, h$. Hence, the intersection of $\overline{\mathbb{A}}$ with the affine open subset $y_0 \neq 0$, or equivalently with $\{x \in \mathbb{P}^{2h+1} : \langle x, e_0 \rangle \neq 0\}$, is exactly \mathbb{A} . So it suffices to show that the image of \mathcal{D} in \mathbb{P}^{2h+1} is contained in the affine open subset $\{x \in \mathbb{P}^{2h+1} : \langle x, e_0 \rangle \neq 0\}$. Equivalently, we must show that there does not exist an x in the image of \mathcal{D} such that $\langle x, e_0 \rangle = 0$.

To see this last statement, suppose that such an x did exist. Let B be any minimal real parabolic subgroup of G contained in P and let K_0 be any maximal compact subgroup of $G(\mathbb{R})$ for which the Iwasawa decomposition $G(\mathbb{R}) = B(\mathbb{R})K_0$ holds. Then $G(\mathbb{R}) = P(\mathbb{R})K_0$ as well. It follows that $P(\mathbb{R})$ acts transitively on \mathcal{D} and preserves the condition that $\langle x, e_0 \rangle = 0$. But then the image of \mathcal{D} would be

contained in $\{x \in \mathbb{P}^{2h+1} : \langle x, e_0 \rangle = 0\}$. This contradicts the fact that \mathcal{D} is open in $\tilde{\mathcal{D}}$. \square

Remark 6.6. If φ is an arbitrary cubic polynomial defined over k and \hat{Z} is defined by Equation 4.12, then $\text{Aut}(\hat{Z})$, the group of symplectic automorphisms of \mathbb{P}^{2h+1} preserving \hat{Z} , is an affine group scheme defined over k . In particular, in the Hermitian symmetric case, if φ is defined over k , then G can be defined over k as well.

The compact duals in the weight three tube domain cases are connected to homogeneous Legendrian submanifolds via the following theorem (e.g. [Muk98], [LM07]).

Theorem 6.7. *The homogeneous Legendrian varieties $X = G(\mathbb{C})/P(\mathbb{C})$ are as follows:*

- (i) *a linear embedding $\mathbb{P}^h \subseteq \mathbb{P}^{2h+1}$ for $G(\mathbb{C}) = \text{SL}(h+3, \mathbb{C})$.*
- (ii) *the Segre embedding $\mathbb{P}^1 \times Q_{h-1} \subseteq \mathbb{P}^{2h+1}$ (with Q_h a quadric) for $G(\mathbb{C}) = \text{SL}(2, \mathbb{C}) \times \text{SO}(n, \mathbb{C})$;*
- (iii) *the twisted cubic $\mathbb{P}^1 \subseteq \mathbb{P}^3$ for $G(\mathbb{C}) = \text{SL}(2, \mathbb{C})$;*
- (iv) *the subexceptional series corresponding to A_5 , C_3 , D_6 , and E_7 respectively (these are discussed in detail in [LM01]).*

Note also that the homogeneous Legendrian varieties correspond precisely to the maximal Hermitian VHS of Calabi–Yau threefold type (see Corollary 2.34 and Remark 2.35). Specifically, the four examples of Gross [Gro94] (cf. Corollary 2.34(i)) correspond to Case (iv) in the above theorem. The remaining cases arise as follows: Case (iii) corresponds to $\text{Sym}^3 V$, where V is the canonical weight one variation of elliptic curve type over the upper half plane \mathfrak{H} (see Corollary 2.34(ii)). Case (ii) corresponds to $\mathfrak{H} \times \mathcal{D}$, where \mathcal{D} is a Type IV_n symmetric space corresponding to variations of $K3$ type (see Corollary 2.34(iv)). Case (i) corresponds to the unit ball as discussed elsewhere in this paper (e.g. Corollary 2.34(iii)).

Remark 6.8. Some examples of non-homogeneous Legendrian varieties have been given by Landsberg–Manivel [LM07] and Buczyński [Buc08]. While these examples give interesting horizontal subvarieties of period domains \mathbf{D} of CY threefold type, they will be stabilized by small groups and thus will not occur as images of period maps (compare with Theorem 1.4 and Theorem 4.16).

Remark 6.9. There is a further connection between the compact duals and Severi varieties (see [Muk98], [Bai00], [LM01]). By a theorem of Zak (see [LVdV84]) there are precisely 4 smooth projective varieties $X \subseteq \mathbb{P}^n$ with $3 \dim X = 2(n-2)$ such that the secant variety $\text{Sec}(X)$ is not (as expected) \mathbb{P}^n . In fact, $\text{Sec}(X) \subseteq \mathbb{P}^n$ is an irreducible cubic hypersurface with large symmetry group. The 4 examples (of dimensions 2, 4, 8, 16) are:

- (1) *The Veronese surface: $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$;*
- (2) *The Segre embedding: $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$;*
- (3) *The Plücker embedding: $G(2, 6) \hookrightarrow \mathbb{P}^{14}$;*
- (4) *The exceptional case: $X \hookrightarrow \mathbb{P}^{26}$, the orbit of the highest weight vector for a 27-dimensional representation of E_6 .*

Here the cases (1)–(4) correspond to the 4 tube domains. In all cases, X is a compact Hermitian symmetric space for the complex form $G'_\mathbb{C}$ of the derived group

of K , the secant variety $\text{Sec}(X)$ is the cubic $C = V(\varphi) \subseteq \mathbb{P}^{h-1}$, and X is the singular locus of $\text{Sec}(X)$.

6.2. The case where the domain is not of tube type. We begin with the analogue of the Landsberg–Manivel picture in this case. Here, if $\mathfrak{g}_{\mathbb{C}}$ is the complex Lie algebra of G and we write $\mathfrak{k}_{\mathbb{C}} \cong \mathfrak{g}'_{\mathbb{C}} \oplus \mathfrak{z}$ as before, then as in the previous case $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{k}_{\mathbb{C}} \oplus W_+ \oplus W_-$, where W_{\pm} are irreducible representations of $\mathfrak{k}_{\mathbb{C}}$ which are minuscule representations of $\mathfrak{g}'_{\mathbb{C}}$, and W_- is the dual of W_+ ; as before $W_{\mp} = \mathfrak{p}_{\pm}$. The minuscule representation V_+ of $\mathfrak{g}_{\mathbb{C}}$ splits as a $\mathfrak{k}_{\mathbb{C}}$ -module as

$$V_+ \cong \underline{\mathbb{C}} \oplus W_+ \oplus W_0,$$

and hence by duality (using $V_- = V_+^{\vee}$ as in Section 2)

$$V_- \cong \underline{\mathbb{C}} \oplus W_- \oplus W_0^{\vee}.$$

Here $\mathfrak{g}'_{\mathbb{C}}$ acts in the standard way on W_{\pm} , W_0 , and W_0^{\vee} and trivially on the two factors $\underline{\mathbb{C}}$, and the center \mathfrak{z} acts diagonally with weights as described in Section 2 (esp. §2.2). As before, $V_{\mathbb{C}} = V_+ \oplus V_-$ has a natural real structure.

The main new feature in the non-tube case is that there exists a $G'_{\mathbb{C}}$ -invariant bilinear form $B: W_+ \otimes W_+ \rightarrow W_0$ or dually $B^*: W_0^{\vee} \otimes W_+ \rightarrow W_-$. For example, in case G is of type E_6 , then $\mathfrak{g}'_{\mathbb{C}}$ is of type D_5 , W_{\pm} are the two 16-dimensional spin representations of $\mathfrak{g}'_{\mathbb{C}}$, $W_0 \cong W_0^{\vee}$ is the standard representation of dimension 10, and B^* is Clifford multiplication. The unit ball case ($G = \text{SU}(1, n)$) is the special case where $W_0 = 0$, $B = 0$, and W_+ is the standard representation of $\text{U}(n)$.

In the above notation, the natural nilpotent action of $\mathfrak{p}_- = W_+$ on V is given by

$$N_w(s, v, e) = (0, sw, B(v, w)),$$

with $N_w^2 \neq 0$, $N_w^3 = 0$. The dual action on $V_- \cong W_0^{\vee} \oplus W_- \oplus \underline{\mathbb{C}}$ is then

$$N_w(e, \xi, t) = (0, B^*(e, w), \langle \xi, w \rangle).$$

Hence, on V , $N_w^2(s, v, e) = (0, 0, sB(w, w))$ and

$$\exp N_w(s, v, e) = (s, v + sw, e + B(v, w) + \frac{s}{2}B(w, w)).$$

In particular

$$\exp N_w(1, 0, 0) = (1, w, \frac{1}{2}B(w, w)),$$

and this is the explicit complex description of the horizontal subvariety Z in this notation (as opposed to that of Equation (5.5)).

In summary, we have shown the following:

Theorem 6.10. *Let $V_{\mathbb{C}} = V_+ \oplus V_-$ be a weight three Hermitian variation of Hodge structure of complex type. Then there exists a complex basis of V_+ and V_- such that the variation of Hodge structure is described by Equation (5.5), where the q_k are homogenous quadratic polynomials.*

Remark 6.11. There are also rationality statements in case G and P are defined over a subfield k of \mathbb{R} and V_{\pm} are defined over an imaginary quadratic extension K of k .

Remark 6.12. In the above notation, we have

- (1) $N_w \neq 0 \iff w \neq 0$.
- (2) If $w \neq 0$, then $\text{Ker } N_w = \{(0, v, e) : B(v, w) = 0\}$.
- (3) $N_w^2 \neq 0 \iff B(w, w) \neq 0$.

6.3. An example. As already mentioned (see Remark 2.35, Remark 5.6, and Equation (5.7)), most of the complex cases can be embedded into maximal Hermitian VHS (of real type). Thus, we consider the following situation:

$$\mathcal{D}' \hookrightarrow \mathcal{D} \hookrightarrow \mathbf{D},$$

where \mathcal{D}' and \mathcal{D} are Hermitian symmetric domains with a holomorphic equivariant embedding $\mathcal{D}' \subset \mathcal{D}$ (as classified by Satake [Sat65] and Ihara [Iha67]), and $\mathcal{D} \hookrightarrow \mathbf{D}$ is a maximal horizontal embedding in a period domain \mathbf{D} of Calabi–Yau threefold type. We assume \mathcal{D} is of tube type, while \mathcal{D}' is not. Clearly, $\mathcal{D}' \hookrightarrow \mathbf{D}$ is a horizontal embedding. It is interesting to compare (via restriction from \mathcal{D} to \mathcal{D}') the description of the embedding of \mathcal{D} given by Theorem 6.5 with that of \mathcal{D}' given by Theorem 6.7. We will discuss this below for the two most interesting cases in the Satake–Ihara classification, both cases corresponding to maximal subdomains in the exceptional domain EVII.

6.3.1. EIII \hookrightarrow EVII. As already noted, the minuscule 27-dimensional representation for E_6 decomposes as a $\text{Spin}(10)$ -module into $\underline{\mathbb{C}} \oplus W_+ \oplus W_0$, where W_+ is one of the half spin representations, of dimension 16, and W_0 is the standard representation, of dimension 10. The procedure of Remark 5.6 defines a 26-dimensional variation of Hodge structure associated to a cubic form Φ_0 , which can be thought of intrinsically as induced from the trilinear form $W_+ \otimes W_0 \otimes W_- \rightarrow \mathbb{C}$ given by Clifford multiplication. On the other hand, the description of the EVII domain is given by Equation (4.12) with Φ the Cartan cubic, the unique E_6 -invariant cubic polynomial on W , where W is a minuscule E_6 -representation (compare (6.4)). The precise relationship between the equations realizing the EIII domain as a horizontal subvariety of CY type (based on Clifford multiplication) and those for the EVII domain (based on the Cartan cubic) is then as follows:

Proposition 6.13. *With notation as above, Φ_0 is the restriction of the Cartan cubic Φ to a hyperplane.*

Proof. Perhaps the most natural way to do so is via del Pezzo surfaces: view the root system for E_6 as the primitive cohomology of X_6 , the blowup of \mathbb{P}^2 at 6 general points, and let h and e_1, \dots, e_6 be the classes of the pullbacks of the hyperplane class in \mathbb{P}^2 and the 6 exceptional divisors, and view the root system for D_5 as lying in the span of h, e_1, \dots, e_5 . The weights of one of the minuscule representations of E_6 correspond to lines on a cubic surface, and three weight spaces pair nontrivially under the cubic \iff the lines sum to a hyperplane section. For D_5 , the weights for W_+ can be taken to correspond to lines on a degree 4 del Pezzo surface X_5 , those for W_0 correspond to pencils of conics, and those for W_- to linear systems of twisted cubics on X_5 , with a line and a twisted cubic pairing nontrivially \iff their sum is a hyperplane section. Then viewing Clifford multiplication as a pairing $W_+ \otimes W_+ \rightarrow W_0$, two weight spaces corresponding to lines ℓ_1 and ℓ_2 pair nontrivially $\iff \ell_1 \cdot \ell_2 = 1$, and in this case the corresponding weight space is the pencil of conics $|\ell_1 + \ell_2|$; equivalently, a line and a conic pair nontrivially \iff their sum is a hyperplane section. To relate this to the Cartan cubic, note that there are 11 lines on X_6 which do not correspond to lines on X_5 . Ten of these correspond to pencils of conics on X_5 : if $|C|$ is such a pencil, there is a unique element $C_0 \in |C|$ passing through the point corresponding to the 6th blowup and the new line is the proper transform of C_0 . The remaining line is e_6 . In this way, we can identify the

restriction of the Cartan cubic to the sum of all of the weight spaces except for that spanned by e_6 (at least over \mathbb{C} , and being somewhat careless about possible scalings of the factors) with Φ_0 . \square

6.3.2. $I_{2,6} \hookrightarrow \text{EVII}$. If we look instead at the subspace of the EVII Hermitian symmetric space corresponding to $\text{SU}(2, 6)$ (i.e. of type $I_{2,6}$), then we see a somewhat different picture. Here, if V_0 is the standard representation for $\text{SU}(2, 6)$, then V_0 decomposes as a representation for the maximal compact subgroup $\text{S}(\text{U}(2) \times \text{U}(6))$ as $W_1 \oplus W_2$, where $\dim W_1 = 2$ and $\dim W_2 = 6$, in the obvious way. Then the representation $V = \bigwedge^2 V_0$ decomposes as $\mathbb{C} \oplus (W_1 \otimes W_2) \oplus \bigwedge^2 W_2$, where $\dim W_1 \otimes W_2 = 12$ and $\dim \bigwedge^2 W_2 = 15$. The recipe of Remark 5.6 constructs the cubic form Φ_0 on $(W_1 \otimes W_2) \oplus \bigwedge^2 W_2^\vee$ given as follows: given $x_1, x_2 \in W_1 \otimes W_2$, write $x_1 \wedge x_2$ for the (symmetric) pairing $(W_1 \otimes W_2) \otimes (W_1 \otimes W_2) \rightarrow (\bigwedge^2 W_1) \otimes (\bigwedge^2 W_2) \cong \bigwedge^2 W_2$, and define the trilinear form $C_0(x_1, x_2, \xi) = \xi(x_1 \wedge x_2)$. Then C_0 corresponds to a cubic for Φ_0 on the 27-dimensional vector space $(W_1 \otimes W_2) \oplus \bigwedge^2 W_2^\vee$. However, Φ_0 is **not** the right choice to be the Cartan cubic. Instead we must modify Φ_0 (as discussed at the end of Remark 5.6), by adding a term Φ_1 , where Φ_1 is the homogeneous cubic corresponding to the symmetric trilinear form C_1 , only nonzero on the $\bigwedge^2 W_2^\vee$ summand, defined by

$$C_1(\xi_1, \xi_2, \xi_3) = \xi_1 \wedge \xi_2 \wedge \xi_3 \in \bigwedge^3 \bigwedge^2 W_2^\vee \cong \mathbb{C}.$$

One can then check that, again over \mathbb{C} and with some care as to the scalings of the weight spaces, the cubic $\Phi_0 + \Phi_1$ is in fact the Cartan cubic.

7. CONCLUDING REMARKS

In this section, we restrict to the weight three tube domain case. Our goal is to identify interesting Hodge theoretical loci in terms of the Hermitian symmetric space \mathcal{D} or the projective geometry of the cubic φ .

7.1. The intermediate Jacobian locus. The most interesting locus of Hodge tensors from a geometric point of view is that where the intermediate Jacobian $JV = (V^{3,0} \oplus V^{2,1})/\Lambda$ is isogenous to a product $J_1 \times J_2$ where J_1 is a polarized abelian variety. Equivalently, there is a symplectic direct sum decomposition over \mathbb{Q} : $V = V_1 \oplus V_2$, where V_1 and V_2 are nonzero sub-Hodge structures of V with $V_1^{3,0} = 0$. We shall refer to the set of all such points in \mathcal{D} as the *intermediate Jacobian locus*.

Lemma 7.1. *Let $\mathcal{D} \hookrightarrow \mathbf{D}$ be a horizontal subvariety of a period domain \mathbf{D} which is of Hermitian type (cf. Definition 2.1). Then the locus $Z \subset \mathcal{D}$ of points for which the associated Hodge structure is decomposable is a union of Hermitian symmetric sub-domains each embedded holomorphically and equivariantly into \mathcal{D} (and thus horizontally into \mathbf{D}).*

Proof. Since the decomposability of the Hodge structure can be expressed in terms of the existence of special Hodge tensors, Z will be a union of Noether–Lefschetz loci in the sense of [GGK11, §II.C]. Each component of a Noether–Lefschetz locus coincides with a component of some Mumford–Tate domain $D_{M'}^\circ$ ([GGK11, Theorem II.C.1]). Clearly, the embedding $D_{M'}^\circ \subset \mathcal{D}$ is holomorphic and horizontal and it corresponds to a specialization of Mumford–Tate groups, i.e. if M is the generic

Mumford–Tate corresponding to \mathcal{D} , then $M' \subseteq M$ and M' is the Mumford–Tate group of some special point (with respect to the rational structure) in \mathcal{D} . Since $\mathcal{D} \hookrightarrow \mathbf{D}$ is a horizontal embedding, $D_{M'}^\circ \hookrightarrow \mathbf{D}$ is also horizontal. Thus, as already argued in the proof of Theorem 1.4, it follows (from [Del79], [Mil11b, Theorem 7.9]) that $D_{M'}^\circ$ is in fact a Hermitian symmetric domain with a holomorphic, horizontal and equivariant embedding into \mathbf{D} . \square

According to the lemma, the possible intermediate Jacobian loci in our situation, i.e. Hermitian VHS \mathcal{V} of CY type over \mathcal{D} , are given by subdomains $\mathcal{D}' \hookrightarrow \mathcal{D}$. Except in very special cases, the induced VHS $\mathcal{V}|_{\mathcal{D}'}$ will split into two components $\mathcal{V}_1 \oplus \mathcal{V}_2$, one of which (\mathcal{V}_2 with our convention) will be of Calabi–Yau type, and thus classified by our results. Conversely, suppose that $\mathcal{D}' \subseteq \mathcal{D}$ is a positive dimensional subdomain, corresponding to an inclusion of \mathbb{Q} -algebraic groups $G' \subseteq G$. If the restriction of the representation $\rho: G \rightarrow \mathrm{Sp}(V)$ is a reducible representation of G' over \mathbb{Q} , then the variation of Hodge structure over \mathcal{D}' splits over \mathbb{Q} and the corresponding intermediate Jacobians acquire abelian variety factors. The possibilities for $\mathcal{D}' \subseteq \mathcal{D}$ have been tabulated by Satake [Sat65] and Ihara [Iha67] as we have already mentioned (e.g. Remark 2.35). An interesting example in this set-up is the case of the embedding of the exceptional domains EIII (associated to E_6) into EVII (associated to E_7) discussed in §6.3.1. In this example the restriction of the VHS of weight 3 CY type over EVII with Hodge numbers $(1, 27, 27, 1)$ will decompose over EIII into a VHS of CY type with Hodge numbers $(1, 26, 26, 1)$ and a Tate twist of a VHS of elliptic curve type. In fact, one can see that both VHS will have weak CM by the same imaginary quadratic field $\mathbb{Q}[\sqrt{-d}]$ (compare to Section 3, especially Corollary 3.3). On the other hand, the restriction of the VHS over the EVII domain to the subdomain of type $I_{2,6}$ remains irreducible over \mathbb{R} , and so the $I_{2,6}$ subdomain is not part of the intermediate Jacobian locus.

While all of the intermediate Jacobian loci in $\mathcal{D} \hookrightarrow \mathbf{D}$ of Hermitian type are given by subdomains \mathcal{D}' , to classify all the possibilities, it does not suffice to consider the groups G and G' (notation as in the preceding paragraph). Instead one has to consider the full (generic) Mumford–Tate groups M and M' and to analyze the possible specializations $M' \subset M$ (recall that G is the derived subgroup of M). Even when starting with a Hermitian VHS satisfying our convention 2.5, its restriction to intermediate Jacobian loci (or more generally Noether–Lefschetz loci) typically will not in general satisfy 2.5. For instance, note that one of the simplest types of Noether–Lefschetz sublocus is that cut out by endomorphisms or equivalently Hodge tensors of height 2 (see [GGK11, p. 52–53]). This leads to the consideration of Hodge structures with weak real or complex multiplication as discussed in Section 3, which in turn leads to involved Galois theoretic arguments (see [GGK11, Chapter VI, e.g. §VI.D]) whose analysis would take us too far afield. Here we will only make some remarks from the perspective of equations defining the horizontal subvariety and apply this to the 1-dimensional case (i.e. $h^{2,1} = 1$). The 1-dimensional case can be understood also from a group theoretic perspective via the results of Green–Griffiths–Kerr (esp. [GGK11, Theorem VII.F.1]).

Remark 7.2. Note that the consideration of Mumford–Tate groups allows one to view also points $z_0 \in \mathbf{D}$ as being of Hermitian type (in the sense considered in this paper). Namely, we say that a point $z_0 \in \mathbf{D}$ is of Hermitian type iff the Mumford–Tate group of z_0 is abelian, i.e. a torus (since it is connected), and thus G will be

trivial. Equivalently, the Hodge structure associated to z_0 will be of CM type (see [GGK11, §V.B]).

7.1.1. The equations cutting out the Noether–Lefschetz locus of type $E \times J_2$. Let us consider the very concrete case where the abelian variety factor J_1 is an elliptic curve E . In this case, given two vectors $\alpha, \beta \in V(\mathbb{Q})$ such that $\langle \alpha, \beta \rangle \neq 0$, the condition that α and β span a two-dimensional sub-Hodge structure of V is the condition that $\alpha \wedge \beta$ is orthogonal to $F^4 \bigwedge^2 V$ for the symmetric pairing $\langle \cdot, \cdot \rangle$ on $\bigwedge^2 V$ induced by the symplectic form on V . Note that

$$F^4 \bigwedge^2 V = (V^{3,0} \otimes V^{2,1}) \oplus (V^{3,0} \otimes V^{1,2}) \oplus \bigwedge^2 V^{2,1}.$$

We will also want the condition that the non-zero Hodge numbers occurring in $\text{span}\{\alpha, \beta\}$ are $h^{2,1}$ and $h^{1,2}$ (rather than $h^{3,0}$ and $h^{0,3}$).

Lemma 7.3. *Let ω be a generator for $V^{3,0}$. Then the following conditions are equivalent:*

- (i) $\alpha \wedge \beta$ is orthogonal to $F^4 \bigwedge^2 V$ and the non-zero Hodge numbers occurring in $\text{span}\{\alpha, \beta\}$ are $h^{2,1}$ and $h^{1,2}$.
- (ii) $\langle \alpha, \omega \rangle = \langle \beta, \omega \rangle = 0$ and $\alpha \wedge \beta$ is orthogonal to $\bigwedge^2 V^{2,1}$.
- (iii) $\langle \alpha, \omega \rangle = \langle \beta, \omega \rangle = 0$ and $V^{2,1} \cap (\mathbb{C} \cdot \alpha + \mathbb{C} \cdot \beta) \neq \{0\}$.

Proof. Clearly, if $\langle \alpha, \omega \rangle = \langle \beta, \omega \rangle = 0$, then $\alpha \wedge \beta$ is orthogonal to $(V^{3,0} \otimes V^{2,1}) \oplus (V^{3,0} \otimes V^{1,2})$, so that if $\alpha \wedge \beta$ is orthogonal to $\bigwedge^2 V^{2,1}$, it is orthogonal to $F^4 \bigwedge^2 V$ as well. Conversely, if $\alpha \wedge \beta$ is orthogonal to $F^4 \bigwedge^2 V$, it is orthogonal to $\bigwedge^2 V^{2,1}$. Moreover, the condition that the non-zero Hodge numbers occurring in $\text{span}\{\alpha, \beta\}$ are $h^{2,1}$ and $h^{1,2}$ implies that $\langle \alpha, \omega \rangle = \langle \beta, \omega \rangle = 0$. This shows the equivalence of the first two conditions, and the equivalence of the second and third is also straightforward. \square

7.1.2. The 1-dimensional case. To illustrate the above discussion, we consider the 1-dimensional case, i.e. $h = \dim V^{2,1} = 1$. Let \mathcal{V} be a Hermitian VHS of weight 3 and CY type over the upper half-plane \mathfrak{H} . We assume that the Hodge structure \mathcal{V}_z at the general point $z \in \mathfrak{H}$ is irreducible, and ask: for which special points of $s \in \mathfrak{H}$ is the Hodge structure $V = \mathcal{V}_s$ decomposable? In this situation $V = V_1 \oplus V_2$, where $V_1 = V_1^{2,1} \oplus V_1^{1,2}$ is a Tate twist of the Hodge structure of an elliptic curve E_1 , and $V_2 = V_2^{3,0} \oplus V_2^{0,3}$, which is formally a double half-twist of the Hodge structure of an elliptic curve E_2 (see (2.8). However, for V_2 to be defined over \mathbb{Q} , one needs the corresponding elliptic curve to have CM. In this set-up, there are two distinct horizontal embeddings of \mathfrak{H} into the period domain \mathbf{D} , one corresponding to $\text{Sym}^3 W$, where W is the Hodge structure of an elliptic curve (Corollary 2.34(ii)), and one corresponding to the complex ball (Corollary 2.34(iii), case $I_{1,n}$ for $n = 1$). We will discuss only the former case here. Thus, we assume $\mathfrak{H} \hookrightarrow \mathbf{D}$ via Sym^3 of the standard representation of $\text{SL}(2)$.

From the point of view of Mumford–Tate domains, it is easy to see which of the points of $\mathfrak{H} \hookrightarrow \mathbf{D}$ are special in the sense discussed above. Specifically, since the generic Mumford–Tate group in this situation is $\text{SL}(2)$, the only possibility for Mumford–Tate groups at special points of \mathfrak{H} is $\text{U}(1)$. Then one can show that

Proposition 7.4. *With notation as above. The special points of \mathfrak{H} correspond precisely to the case when the Hodge structure V acquires weak CM by an imaginary quadratic field $\mathbb{Q}[\sqrt{-d}]$ (see §3).*

Moreover, one can show that the elliptic curves E_1 and E_2 (corresponding to the components V_1 and V_2 of V) have CM by $\mathbb{Q}[\sqrt{-d}]$. This follows for instance from [GGK11, Theorem VII.F.1] (cases (v) and (xii)). Here we will prove the claim 7.4 by using the explicit description of the embedding $\mathfrak{H} \hookrightarrow \mathbf{D}$ given by (4.5).

Proof. The condition of Lemma 7.3 that $\alpha \wedge \beta$ is orthogonal to $\bigwedge^2 V^{2,1}$ is vacuous. In this case, we can write the variation of the generator of $V^{3,0}$ as $\omega(z) = -cz^3e_0 + 3cz^2e_1 + zf_1 + f_0$ for z in the upper half plane, where we assume that $c \in \mathbb{Q}$, $c \neq 0$. Setting

$$\begin{aligned}\alpha &= \alpha_0e_0 + \alpha_1e_1 + \alpha'_1f_1 + \alpha'_0f_0; \\ \beta &= \beta_0e_0 + \beta_1e_1 + \beta'_1f_1 + \beta'_0f_0\end{aligned}$$

leads to two equations

$$\begin{aligned}-c\alpha'_0z^3 + 3c\alpha'_1z^2 - \alpha_1z - \alpha_0 &= 0; \\ -c\beta'_0z^3 + 3c\beta'_1z^2 - \beta_1z - \beta_0 &= 0,\end{aligned}$$

where the coefficients are rational, and hence after possibly eliminating the constant term to an equation of the form $zP(z) = 0$, where $P(z)$ is a quadratic polynomial with rational coefficients. Thus z lies in an imaginary quadratic field. Conversely, if $P(t)$ is an irreducible quadratic polynomial with rational coefficients and z is a root of P in the upper half plane, choose two rational numbers r_1, r_2 and consider the cubic polynomials $Q_1(t) = (t + r_1)P(t)$ and $Q_2(t) = (t + r_2)P(t)$. The coefficients of Q_1, Q_2 determine rational vectors α, β such that $\langle \alpha, \omega \rangle = \langle \beta, \omega \rangle = 0$, and a computation shows that, up to a factor of $\frac{1}{3}$, $\langle \alpha, \beta \rangle = r_1 - r_2$. Hence, if r_1 and r_2 are distinct, we produce a two dimensional subspace of $V(\mathbb{Q})$ with the desired properties. Here, the choice of r_1 and r_2 is essentially equivalent to the choice of a \mathbb{Q} -basis for the span of α and β . \square

Remark 7.5. In the non-algebraic case of Equation 4.5, $\omega(z)$ is given by

$$\omega(z) = \left(\varphi(z) - \frac{1}{2}z \frac{d\varphi}{dz} \right) e_0 + \frac{1}{2} \frac{d\varphi}{dz} e_1 + zf_1 + f_0,$$

where φ is only assumed to be a holomorphic function of z . The equations $\langle \alpha, \omega \rangle = \langle \beta, \omega \rangle = 0$ become two equations of the form

$$\begin{aligned}c_1\varphi(z) + \ell_1(z) \frac{d\varphi}{dz} + m_1(z) &= 0; \\ c_2\varphi(z) + \ell_2(z) \frac{d\varphi}{dz} + m_2(z) &= 0,\end{aligned}$$

where the $c_i \in \mathbb{Q}$ and the ℓ_i, m_i are degree one polynomials of z with coefficients in \mathbb{Q} . In particular, both $\varphi(z)$ and $d\varphi(z)/dz$ lie in $\mathbb{Q}(z)$ (and are of a very special form there). In case φ is a cubic and z is imaginary quadratic, the existence of one equation of the above type leads to a second such. It would be interesting to find examples of transcendental functions φ and points z which have analogous properties.

Remark 7.6. Green–Griffiths–Kerr [GGK11, §VII.F] (especially Theorem VII.F.1) discuss in detail the Noether–Lefschetz loci in the case of weight 3 CY Hodge structures with $h^{2,1} = 1$. Specifically, the cases (iii) and (v) of [GGK11, Theorem VII.F.1] correspond to two irreducible case of Hermitian VHS with $h^{2,1} = 1$ in our classification: the ball case and the Sym^3 case respectively. The cases (viii)–(xii) of [GGK11, Theorem VII.F.1] classify the possible Mumford–Tate groups for Hodge structures that decompose as $V = V_1 \oplus V_2$. Since we are considering Noether–Lefschetz subloci of Hermitian symmetric domains, the only relevant cases for us are (xi) and (xii). The most interesting situation, the case (v) specializing to the case (xii), was discussed above (Proposition 7.4). For some geometric situations that lead to these two cases (essentially some special cases of the Borcea–Voisin construction) we refer the reader to [GGK11, p. 201–202].

7.2. The Baily–Borel compactification and monodromy. We keep the notation of §6.1, so that $V = \mathbb{C} \oplus W \oplus W^\vee \oplus \mathbb{C}$ and $B: \text{Sym}^2 W \rightarrow W^\vee$ is the bilinear form. We begin by analyzing the following general situation: \mathcal{V} is a weight three \mathbb{Q} -VHS of CY type over Δ^* with general fiber V is a complex vector space with a rational structure and a nondegenerate symplectic form defined over \mathbb{Q} . Let $N = \log T$, where $T \in G(\mathbb{Z})$ is the monodromy matrix, which we assume to be unipotent. In particular, N is a rational nilpotent matrix preserving the symplectic form such that $N^4 = 0$. Let F^\bullet be the limiting Hodge filtration and W_\bullet be the corresponding monodromy weight filtration. Thus $(V, F^\bullet, W_\bullet)$ is a mixed Hodge structure satisfying the usual properties: $N: W_k/W_{k-1} \rightarrow W_{k-2}/W_{k-3}$ is a morphism of Hodge structures of type $(-1, -1)$, and $N^k: W_{3+k}/W_{3+k-1} \rightarrow W_{3-k}/W_{3-k-1}$ is an isomorphism. Note that $N^4 = 0$.

Definition 7.7. We shall say that the limiting mixed Hodge structure is of *Type IV* if $N^3 \neq 0$, of *Type III* if $N^3 = 0$, $N^2 \neq 0$, and of *Type II* if $N^2 = 0$, $N \neq 0$. (Type I is the condition that $N = 0$.)

We discuss the various possibilities below, first in general and then for the Hermitian symmetric tube domain case. Here, we shall use the fact (Remark 6.2) that, in the notation of Section 5, every nilpotent N of the type we are considering is conjugate in $G(\mathbb{C})$ to an element of \mathfrak{p}_- (or equivalently that every unipotent $T \in G(\mathbb{R})$ is conjugate in $G(\mathbb{C})$ to an element of P_- to analyze the possibilities for the monodromy weight filtration over \mathbb{C} and relate them to the form B . Of course, a more careful analysis would proceed via the fine structure of Hermitian symmetric spaces and would give information about the real structure. For the three classical cases, and especially for $\text{Sp}(6, \mathbb{R})$ and $\text{SU}(3, 3)$, one can work out all of the statements below directly.

Type IV: $N^4 = 0$, $N^3 \neq 0$ (or *maximal unipotent monodromy*). The Calabi–Yau condition implies that $F^3 \rightarrow W_6/W_5$ is an isomorphism and that $W_5/W_4 = 0$. In particular, N has a unique Jordan block of length 4 and no Jordan block of the length 3. The remaining Jordan blocks must be of length 2 or 1. In terms of the weight filtration, this says that W_4/W_3 is a Hodge structure of pure type $(2, 2)$ and $N: W_4/W_3 \rightarrow W_2/W_1$ is an isomorphism of type $(-1, -1)$, so that W_2/W_1 is pure of type $(1, 1)$. This leaves open the possibility that W_3/W_2 is nonzero, whose nonzero summands are of Hodge type $(2, 1)$ and $(1, 2)$, corresponding to length 1 Jordan blocks of N . It is easy to see that there are no length 1 Jordan blocks $\iff W_3 = W_2 \iff N$ induces an isomorphism from W_4/W_2 to W_2/W_0 .

In the Hermitian case, we can assume that $N = N_w$ corresponds to the action of w on V . Using the explicit description of N_w , we see that Type IV corresponds to $C(w) = \langle B(w, w), w \rangle \neq 0$, and the condition that there are no length 1 Jordan blocks is equivalent to:

Condition 7.8. *If $C(w) \neq 0$, then the linear map B_w defined by $v \in W \mapsto B(w, v) \in W^\vee$ is an isomorphism from W to W^\vee .*

This follows easily, in the EVII case, from the fact that the only E_6 -invariant polynomials on the minuscule representation W are polynomials in the Cartan cubic (here denoted C).

Type III: $N^3 = 0, N^2 \neq 0$. In this case, $V = W_6 = W_5$ and there is a length 3 Jordan block, so that $W_5/W_4 \neq 0$. The Calabi–Yau condition then implies that $\dim W_5/W_4 = 2$ and that up to a Tate twist W_5/W_4 is of elliptic curve type (the nonzero summands are one-dimensional of type $(3, 2)$ and $(2, 3)$). Of course, a similar statement is true for $W_1/W_0 = W_1$, and, since $N^2: W_5/W_4 \rightarrow W_1$ is an isomorphism, N induces an injection from W_5/W_4 to W_3/W_2 . Again, the remaining Jordan blocks must be of length 2 or 1 and the Hodge structure on W_4/W_3 is of pure type $(2, 2)$ and that on W_2/W_1 is pure of type $(1, 1)$. Again, *a priori* there could be length 1 Jordan blocks of N corresponding to the possibility that W_3/W_2 is strictly larger than the 2-dimensional image of N .

In the tube domain case, the condition $N^3 = 0, N^2 \neq 0$ is equivalent to $C(w) = 0$ but $B(w, w) \neq 0$. The condition that there are no length 1 Jordan blocks is equivalent to:

Condition 7.9. *Let L be the hyperplane $\text{Ker } B(w, w) \subseteq W$, let L^* be the hyperplane $w^\perp = \{\xi \in W^\vee : \langle \xi, w \rangle = 0\}$, and let $B_w: L \rightarrow L^*$ be the linear map $B_w(v) = B(w, v)$. (Note that $w \in L$ and that, if $v \in L$, then $B_w(v)(w) = \langle B(w, v), w \rangle = \langle B(w, w), v \rangle = 0$ by definition, so that $B_w(v) \in L^*$.) Then B_w is an isomorphism from L to L^* .*

In the EVII case, this also presumably reduces to a known fact about the Cartan cubic.

Type II: $N^2 = 0, N \neq 0$. In this case, $W_4 = V$, $W_3 = \text{Ker } N$, and $W_2 = \text{Im } N$. Of course, this holds in the Type I case also. There is no Jordan block of length 1 $\iff W_3 = W_2 \iff \text{Ker } N = \text{Im } N$. In the cases of interest to us, we will in fact have $W_3 \neq W_2$, so the mixed Hodge structure will not be of Hodge–Tate type.

In the tube domain case, $N^2 = 0, N \neq 0 \iff B(w, w) = 0$ but $w \neq 0$. In this case,

$$\text{Im } N = \{(0, tw, B(w, v), s) : t, s \in \mathbb{C}, v \in W\};$$

$$\text{Ker } N = \{(0, v, \xi, s) : v \in \text{Ker } B_w, \xi \in w^\perp\}.$$

Thus $\text{Ker } N / \text{Im } N \cong (\text{Ker } B_w / \mathbb{C} \cdot w) \oplus (w^\perp / \text{Im } B_w)$, where as before $B_w(v) = B(w, v)$, we have repeatedly used $B(w, w) = 0$, and the two summands are dual to each other. Note that, by the discussion of Remark 6.9, the image in \mathbb{P}^{h-1} of the set of $w \in W$ such that $B(w, w) = 0$ is a single orbit under $K(\mathbb{C})$, and hence the dimensions of the spaces W_3 and W_2 are independent of the choice of w .

The above does not specify the shape of the mixed Hodge structure. In general, we make the following definition:

Definition 7.10. Let $(V, F^\bullet, W_\bullet)$ be a weight three limiting mixed Hodge structure of Calabi–Yau type ($\dim F^3 = 1$) such that $N^2 = 0$, $N \neq 0$. We say that $(V, F^\bullet, W_\bullet)$ is of *Picard–Lefschetz type* if W_4/W_3 is of pure weight $(2, 2)$, or equivalently if W_3/W_2 is of Calabi–Yau type. Note that, if $(V, F^\bullet, W_\bullet)$ is not of Picard–Lefschetz type, then W_4/W_3 is (up to a Tate twist) of *K3 type*.

In the case of interest to us, Picard–Lefschetz type does not arise (cf. the remark by Deligne in §10 of [Gro94]):

Lemma 7.11. *In the Hermitian symmetric case, with $N^2 = 0$ but $N \neq 0$, the resulting limiting mixed Hodge structure is never of Picard–Lefschetz type.*

Proof. (Sketch.) Clearly $(V, F^\bullet, W_\bullet)$ is of Picard–Lefschetz type $\iff N\omega = 0$, where $F^3 = \mathbb{C} \cdot \omega$, $\iff T\omega = \omega$. On the other hand, up to conjugacy, there is a boundary component F such that $T \in U(F)$ and ω is induced by an element of $\mathcal{D}(F) = U(F)(\mathbb{C}) \cdot \mathcal{D}$ (cf. [AMRT75], §7 of Chapter III). Hence $\omega = S\omega_0$ for some $S \in U(F)(\mathbb{C})$, where ω_0 is induced by an element of \mathcal{D} . Since $U(F)$ is abelian, it follows that $T\omega_0 = \omega_0$. But then ω_0 would be fixed by a nontrivial unipotent element of $G(\mathbb{R})$, contradicting the fact that its stabilizer in $G(\mathbb{R})$ is compact. \square

To tie the above picture in with the general theory of Hermitian symmetric spaces, Korányi and Wolf have classified the boundary components of the Hermitian symmetric spaces [WK65]. In the four cases of irreducible tube domains of real rank three, the boundary components are almost completely specified by the conditions that they be of tube type and of real rank 0, 1, 2. In particular, the real rank 0 boundary components are points, the real rank 1 boundary components are copies of the upper half plane, and the real rank two boundary components are the rank two Hermitian symmetric spaces associated to $\mathrm{SO}(2, 10)$ in case G is of type E_7 , $\mathrm{SO}^*(8) = \mathrm{SO}(2, 6)$ in case G is of type D_5 , $\mathrm{SU}(2, 2) = \mathrm{SO}(2, 4)$ in case G is of type A_5 , and of type $C_2 = B_2$, i.e. $\mathrm{Sp}(4, \mathbb{R}) \cong \mathrm{SO}(2, 3)$, in case G is of type C_3 (all equalities mod the center).

Putting this together, we see that, in the Type IV case, the mixed Hodge structure is of Hodge–Tate type. In the Type III case, $W_2 = W_2/W_0$ is an extension of a weight one Hodge structure by a pure weight two Hodge structure and $W_6/W_3 = W_5/W_3$ is a Tate twist of the dual of W_2 . In this case, presumably all of the information of the mixed Hodge structure is contained in W_2 . In the Type II case, the Hodge structure on W_2 is of *K3 type*, the one on W_4/W_3 is a Tate twist of W_2 , and the one on W_3/W_2 is, up to a Tate twist, the Kuga–Satake construction applied to the weight two Hodge structure on W_2 , because the representation of $\mathrm{SO}(2, k)$ corresponding to W_3/W_2 is essentially a spin or half spin representation. It seems likely that, in all cases, the essential information of the mixed Hodge structure is contained in the two step extension W_3 , and that this extension can be described quite explicitly.

Remark 7.12. In the complex case, the weight three variation of Hodge structure is obtained by reassembling a weight one variation (in the unit ball case) or a weight two variation (in the remaining cases). Thus, we must have $N^2 = 0$ in the unit ball case and $N^3 = 0$ in the remaining cases. Of course, it is easy to see directly in the case of $\mathrm{SU}(1, n)$, or more generally $\mathrm{SU}(p, q)$, that, if P is a maximal real parabolic subgroup corresponding to a zero-dimensional boundary component, U is the unipotent radical of P , and $N \in U$, then $N^2 = 0$, where we view N as acting

on \mathbb{C}^{p+q} in the standard representation (compare Rohde [Roh10], who notes that in the unit ball case the monodromy is never maximal unipotent).

One can also analyze the resulting limiting mixed Hodge structures along the lines of the tube domain case. For example, in the unit ball case, viewing the complex representation $V \cong \mathbb{C} \oplus W$ with dual $W^\vee \oplus \mathbb{C}$, if $N = N_w$ is a nontrivial monodromy matrix, then W_1 and W_4/W_3 are of rank two and pure types $(1, 1)$ $(2, 2)$, respectively, and $W_3/W_2 \cong W/\mathbb{C} \cdot w \oplus w^\perp$, but the Hodge structure on W_3/W_2 is constant. Similar but slightly more involved statements hold in the remaining cases; note here that the real rank one boundary components are unit balls.

REFERENCES

- [AMRT75] A. Ash, D. Mumford, M. Rapoport, and Y. Tai. *Smooth compactification of locally symmetric varieties*. Math. Sci. Press, Brookline, Mass., 1975. Lie Groups: History, Frontiers and Applications, Vol. IV.
- [And92] Y. André. Mumford-Tate groups of mixed Hodge structures and the theorem of the fixed part. *Compositio Math.*, 82(1):1–24, 1992.
- [Bai00] W. L. Baily, Jr. Exceptional moduli problems. II. *Asian J. Math.*, 4(1):1–9, 2000. Kodaira’s issue.
- [BG83] R. L. Bryant and P. A. Griffiths. Some observations on the infinitesimal period relations for regular threefolds with trivial canonical bundle. In *Arithmetic and geometry, Vol. II*, volume 36 of *Progr. Math.*, pages 77–102. Birkhäuser Boston, Boston, MA, 1983.
- [Bor97] C. Borcea. $K3$ surfaces with involution and mirror pairs of Calabi-Yau manifolds. In *Mirror symmetry, II*, volume 1 of *AMS/IP Stud. Adv. Math.*, pages 717–743. Amer. Math. Soc., Providence, RI, 1997.
- [Buc08] J. Buczyński. Hyperplane sections of Legendrian subvarieties. *Math. Res. Lett.*, 15(4):623–629, 2008.
- [Del79] P. Deligne. Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 247–289. Amer. Math. Soc., Providence, R.I., 1979.
- [DK07] I. V. Dolgachev and S. Kondō. Moduli of $K3$ surfaces and complex ball quotients. In *Arithmetic and geometry around hypergeometric functions*, volume 260 of *Progr. Math.*, pages 43–100. Birkhäuser, Basel, 2007.
- [DM86] P. Deligne and G. D. Mostow. Monodromy of hypergeometric functions and nonlattice integral monodromy. *Inst. Hautes Études Sci. Publ. Math.*, 63:5–89, 1986.
- [Fri91] R. Friedman. On threefolds with trivial canonical bundle. In *Complex geometry and Lie theory (Sundance, UT, 1989)*, volume 53 of *Proc. Sympos. Pure Math.*, pages 103–134. Amer. Math. Soc., Providence, RI, 1991.
- [GGK11] M. Green, P. A. Griffiths, and M. Kerr. *Mumford-Tate groups and domains (Their geometry and arithmetic)*. To appear in *Annals of Math Studies.*, 2011.
- [Gro94] B. H. Gross. A remark on tube domains. *Math. Res. Lett.*, 1(1):1–9, 1994.
- [GSvSZ10] R. Gerkmann, M. Sheng, D. van Straten, and K. Zuo. On the monodromy of the moduli space of Calabi-Yau threefolds coming from eight planes in \mathbb{P}^3 . arXiv:1006.5334, 2010.
- [GvG10] A. Garbagnati and B. van Geemen. Examples of Calabi-Yau threefolds parametrised by Shimura varieties. arXiv:1005.0478v1, 2010.
- [Iha67] S. Ihara. Holomorphic imbeddings of symmetric domains. *J. Math. Soc. Japan*, 19:261–302, 1967.
- [Kna02] A. W. Kna. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 2002.
- [KP11] J. Kollár and J. Pardon. Algebraic varieties with semialgebraic universal cover. arXiv:1104.2309, 2011.
- [KW65] A. Korányi and J. A. Wolf. Realization of hermitian symmetric spaces as generalized half-planes. *Ann. of Math. (2)*, 81:265–288, 1965.

- [LM01] J. M. Landsberg and L. Manivel. The projective geometry of Freudenthal’s magic square. *J. Algebra*, 239(2):477–512, 2001.
- [LM07] J. M. Landsberg and L. Manivel. Legendrian varieties. *Asian J. Math.*, 11(3):341–359, 2007.
- [LVdV84] R. Lazarsfeld and A. Van de Ven. *Topics in the geometry of projective space*, volume 4 of *DMV Seminar*. Birkhäuser Verlag, Basel, 1984. Recent work of F. L. Zak, With an addendum by Zak.
- [Mil11a] J. S. Milne. Algebraic Groups, Lie Groups, and their Arithmetic Subgroups. Book available at www.jmilne.org/math/CourseNotes/ALA.pdf, 2011.
- [Mil11b] J. S. Milne. Shimura varieties and moduli. To appear in “Handbook of Moduli” (arXiv:1105.0887), 2011.
- [Moo99] B. Moonen. Notes on Mumford-Tate groups. Preprint available at <http://staff.science.uva.nl/~bmoonen/NotesMT.pdf>, 1999.
- [Muk98] S. Mukai. Simple Lie algebra and Legendre variety. Preprint available at <http://www.kurims.kyoto-u.ac.jp/~mukai/paper/warwick15.pdf>, 1998.
- [O’G06] K. G. O’Grady. Irreducible symplectic 4-folds and Eisenbud-Popescu-Walter sextics. *Duke Math. J.*, 134(1):99–137, 2006.
- [Roh09] J. C. Rohde. *Cyclic coverings, Calabi-Yau manifolds and complex multiplication*, volume 1975 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.
- [Roh10] J. C. Rohde. Maximal automorphisms of Calabi-Yau manifolds versus maximally unipotent monodromy. *Manuscripta Math.*, 131(3-4):459–474, 2010.
- [Sat65] I. Satake. Holomorphic imbeddings of symmetric domains into a Siegel space. *Amer. J. Math.*, 87:425–461, 1965.
- [Sch74] W. Schmid. Abbildungen in arithmetische Quotienten hermitesch symmetrischer Räume. In *Classification of algebraic varieties and compact complex manifolds*, pages 243–258. Lecture Notes in Math., Vol. 412. Springer, Berlin, 1974.
- [SZ10] M. Sheng and K. Zuo. Polarized variation of hodge structures of Calabi-Yau type and characteristic subvarieties over bounded symmetric domains. *Math. Ann.*, 348(1):211–236, 2010.
- [Tre10] T. Trenner. A curvature formula for the complexified index cone of a cubic form. arXiv:1007.2737, 2010.
- [TW11] T. Trenner and P. M. H. Wilson. Asymptotic curvature of moduli spaces for Calabi-Yau threefolds. *J. Geom. Anal.*, 21(2):409–428, 2011.
- [UY11] E. Ullmo and A. Yafaev. A characterization of special subvarieties. To appear in *Mathematika*, 2011.
- [vG01] B. van Geemen. Half twists of Hodge structures of CM-type. *J. Math. Soc. Japan*, 53(4):813–833, 2001.
- [vG08] B. van Geemen. Real multiplication on $K3$ surfaces and Kuga-Satake varieties. *Michigan Math. J.*, 56(2):375–399, 2008.
- [vGI02] B. van Geemen and E. Izadi. Half twists and the cohomology of hypersurfaces. *Math. Z.*, 242(2):279–301, 2002.
- [Voi93] C. Voisin. Miroirs et involutions sur les surfaces $K3$. *Astérisque*, (218):273–323, 1993. Journées de Géométrie Algébrique d’Orsay (Orsay, 1992).
- [Voi99] C. Voisin. *Mirror symmetry*, volume 1 of *SMF/AMS Texts and Monographs*. American Mathematical Society, Providence, RI, 1999. Translated from the 1996 French original by Roger Cooke.
- [WK65] J. A. Wolf and A. Korányi. Generalized Cayley transformations of bounded symmetric domains. *Amer. J. Math.*, 87:899–939, 1965.
- [Zar83] Yu. G. Zarhin. Hodge groups of $K3$ surfaces. *J. Reine Angew. Math.*, 341:193–220, 1983.
- [Zar84] Yu. G. Zarhin. Weights of simple Lie algebras in the cohomology of algebraic varieties. *Izv. Akad. Nauk SSSR Ser. Mat.*, 48(2):264–304, 1984.

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